Morse Genericity and Morse's Theorem for compact smooth manifolds.

Tomas Kojar

MAT 477Y

March 25-th, 2014

Supplementary : Storyline

Parametric-Transversality:
$F(x,s) \pitchfork Z \Rightarrow F_s \pitchfork Z$

Morse Generic

$\exists$ $not\ deg\ V_T$: $\sum_{x \in V_T^{-1}(0)} Ind_p V_T = \chi(M)$

$\sum_{p \in V^{-1}(0)} sign(DV(p))$ homotopy invariant
when V at $p \in V^{-1}(0)$ is not degenerate

Special case Poincare-Hopf:
$\chi(M) = \sum_{p \in V^{-1}(0)} Ind_p(V)$

Non-degenerate zero $\Leftrightarrow$ isolated

Morse Theorem for Morse f on M
$\chi(M) = \sum_{i=0}^{n} (-1)^i cr_i(f)$

Statements of Main Theorems and Lemmas:

1. Constructive proof: Morse functions are dense in $C^{\infty}(M)$;

2. Parametric Transversality: If $F \pitchfork Z \subset N$ for $F : M \times S \to N$ and

$F_s(x) := F(x, s)$, then $F_s \pitchfork Z$ for a.e. s, shortly almost every s ;

3. Construction of vector fielf V_T with $\sum_{p \in V_T^{-1}(0)} Ind_p(V_T) = \chi(M)$;

4. $deg(\partial U_p \ni x \mapsto \frac{V(x)}{|V(x)|}) = sign(det[DV](p))$ at n-d $p \in V^{-1}(0)$;

5. Poincare-Hopf via a homotopy between V_T and a v.f V_1 , e.g. $\bigtriangledown f$.

Regular/Critical points, Transversality, Index, Morse funct

Here $F \in C^\infty(M,N)$, $Z \hookrightarrow N$, map F is **transversal** to Z , shortly $F \pitchfork Z$, when $(DF)(p)T_pM + T_{F(p)}Z = T_{F(p)}N$ for $p \in F^{-1}(Z)$ and if $F(M) \supset Z = \{y\}$ then y is called **regular**. Our $f \in C^\infty(M,\mathbb{R})$, **critical** points $Cr(f) := \{p : df_p = 0\}$ are **non-degenerate**, shortly n-d, when $det(Hess_p(f)) \neq 0$ and $Ind_p(f) := \#$ of negative eigenvalues of $Hess_p(f)$; $0^*_M :=$ 0-section of $T^*M \ni (p, df(p)) =: j^1 f(p)$. Finally, $x = (x_1, ..., x_m)$ are local coordinates on M and f is **Morse** when all $p \in Cr(f)$ are n-d's.

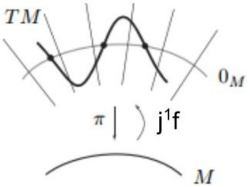
Examples of Morse functions: $f(x) = x_1^2 - x_2^2$ and $f : S^{n-1} \ni x \to x_n$.

Claim: f **is Morse iff** $j^1 f \pitchfork 0_M^*$ **.**

Proof: Say $\psi : M \times \mathbb{R}^m \to \mathbb{R}^m$ is the natural projection. Locally $0_M^* =$

$M \times \{0\} \hookrightarrow M \times \mathbb{R}^m$. Therefore $j^1 f \pitchfork 0_M^* \;\Leftrightarrow\;$ for $p \in (j^1 f)^{-1}(0_M^*)$ linear

maps $\psi D(j^1 f)_p = Hess_p f : T_p M \to \mathbb{R}^m$ are surjective. ■



$j^1 f$ transverse to 0^*_M

Morse property is generic: $\forall \{f_j\}_{0\leq j\leq N} \subset C^\infty(M,\mathbb{R})$, all $T_p^*(M) =$

$Span_{\mathbb{R}}\{df_j(p)\}_{1\leq j} \;\Rightarrow\; f_s(p) := f_0(p) + \sum_{1\leq j} s_j f_j(p)$ is Morse for a.e. s

Proof: Via PTL with $F(p,s) := (p, df_s(p))$ since $F \pitchfork O_M^*$ because maps

$\psi DF_{(x,s)}(\xi,\eta) = \sum_{1\leq j} \eta_j df_j(x) + Hess_x f_s(\xi) \in R^m$ are onto. ■

PTL = Parametric Transversality Lemma: If $F \in C^\infty(M \times S \ , N)$ and

$F \pitchfork Z \subset N$, then $F_s \pitchfork Z$, where $F_s(x) := F(x,s)$, for almost every s .

Main Theorem, Morse: for f Morse $\chi(M) = \sum_{i=0}^n (-1)^i cr_i(f)$, where

$cr_i(f) := \#\{p \in Cr(f) : Ind_p(f) = i\}$ and $m = \dim M$.

Proof of PTL: Let $W := F^{-1}(Z)$ and π be the restriction to W of projection $M \times S \to S$. Using Sard's Thm suffices to show that if b is a regular value for $\pi \;\Rightarrow\; F_b \pitchfork Z$. So, $DF_{(x,b)}(T_{(x,b)}(M \times S)) + T_z Z = T_z N$ for $z := F_b(x) \in Z$ and we assume $D\pi_{x,b} : T_{(x,b)}W \to T_b S$ is onto. Then $\forall\; v_N \in T_z N \;\; \exists \;\; (v_M, v_S) \in T_{(x,b)}(M \times S)$ and $w \in T_x M$ s.th. both $v_N - DF_{(x,b)}(v_M, v_S)$, $DF_{(x,b)}(w, v_S) \in T_z Z$ with $(w, v_S) \in T_{(x,b)} W$.

Then $v_N - D(F_b)_x(v_M - w) = v_N - DF_{(x,b)}[(v_M, v_S) - (w, v_S)] \in T_z Z$ ■

Corollary: Restriction of generic height functions to $M \hookrightarrow \mathbb{R}^n$ are Morse.

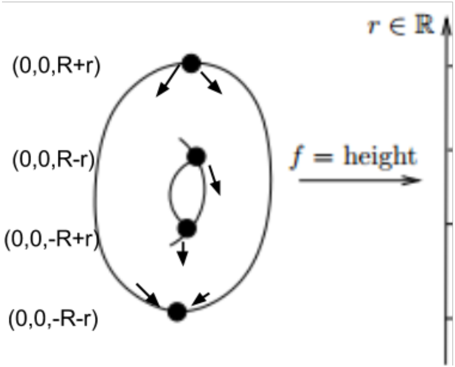
Application. Finding χ visually for surfaces $M_g \hookrightarrow \mathbb{R}^3$ of genus g :

For Morse function $f(x, y, z) = z$

$Ind_{maxima} = 2$, $Ind_{minima} = 0$,

$Ind_{saddle} = 1 \Rightarrow \chi(M_g) = (-1)^0 +$

$(-1)^1 \cdot 2g + (-1)^2 \cdot 1 = 2 - 2 \cdot g$



For a vector field V on

contractible nbd U with

$U \cap V^{-1}(0) = \{p\}$ let

$Ind_p(V) := deg(\partial U \ni x \xmapsto{\phi_{V,p}} \frac{V(x)}{|V(x)|} \in S^{m-1})$. When $det(DV(p)) \neq 0 \Rightarrow$

$\phi_{V,p}$ is homotopic to $x \mapsto \frac{DV(p)x}{|DV(p)x|} \;\Rightarrow\; deg(\phi_{V,p}) = sign(det[DV](p))$.

Lemma: Non-degenerate $p \in Cr(f)$ are isolated.

Indeed, $\bigtriangledown f(p) = 0$ and $det(Hess_p(f)) \neq 0 \stackrel{Inv\ Map\ Thm}{\Rightarrow} \exists$ nbhd U of p

s.th. $\bigtriangledown f|_U$ is a diffeo $\Rightarrow$ $\bigtriangledown f(x) \neq \bigtriangledown f(p) = 0 \; \forall \; x \in U \setminus \{p\}$. ■

Morse Theorem follows from Poincare-Hopf degree Thm:

$\# V^{-1}(0) < \infty$ for vec. fields V on M $\Rightarrow \chi(M) = \sum_{p \in V^{-1}(0)} Ind_p(V)$

Proof: $sign(det[D(\bigtriangledown f)](p)) = (-1)^{Ind_p(f)}$ for f Morse, $p \in Cr(f)$

$\stackrel{P-H\ deg\ Thm}{\Rightarrow} \quad \chi(M) = \sum_{x \in (\bigtriangledown f)^{-1}(0)} Ind_x(\bigtriangledown f) = \sum_i (-1)^i cr_i(f)$. ■

Exists V_T s.th. $Ind(V_T) := \sum_{p\in V_T^{-1}(0)} Ind_p(V_T) = \chi(M)$

and $det([DV_T](p)) \neq 0$ for $p \in V_T^{-1}(0)$. **Construction of vec. f. V_T :**

We start with a triangulation $T(M)$ of M (as in Vitali's talk, page 8).

Denote c_σ the centers and U_σ nbhds of simplexes σ with coordinates (u, v)

centered at c_σ s.th. $c_\sigma \in U_\sigma \not\ni c_\tau$ for simplexes τ , $dim\ \tau \geq dim\ \sigma$ and

$\tau \neq \sigma$; $\{v = 0\} \supset \sigma$, while $dim\{v = 0\} = dim\ \sigma$. On U_σ we define

vector fields $V_\sigma := \bigtriangledown(|u|^2 - |v|^2)$. We then inductively construct V_k on

'small' nbds $\mathcal{U}_k$ of the k-skeletons of $T(M)$, $k \geq 1$, by extending V_{k-1}

from nbhds $\mathcal{U}_{k-1}$ (perhaps shrinking the latter), and set $V_T := V_m$,

$m = dimM$, namely: using the PofU we construct nonnegative C^∞

functions ψ_{k-1} and ϕ_k with supports in $\mathcal{U}_{k-1}$ and $\mathcal{U}_k$ s.th $\psi_{k-1} \equiv 1$

and $\psi_{k-1} + \phi_k \equiv 1$ on nbds of the $(k-1)$ and of the k-skeletons of $T(M)$

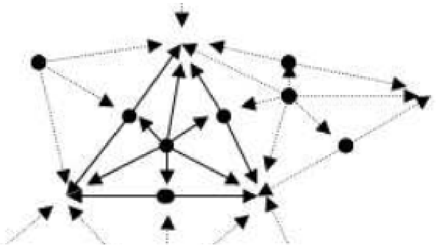
and define $V_k := \psi_{k-1} \cdot V_{k-1} + \phi_k \sum_{\{\sigma : dim\ \sigma = k\}} V_\sigma \ \Rightarrow\ V_T^{-1}(0) = \cup_\sigma \{c_\sigma\}$

(at $p \in \sigma$ with $dim\ \sigma = k$, $\psi_{k-1}(p) \cdot \phi_k(p) \neq 0$ we use that $-V_k(p) \notin$

$\mathbb{R}_+ \cdot \{V_{k-1}(p)\}$) $\Rightarrow Ind_{c_\sigma} V_T = det(DV_T(c_\sigma) = (-1)^{dim\ \sigma}$ for all σ $\Rightarrow$

$Ind(V_T) = \sum_{0\leq k\leq m} (-1)^k s_k =: \chi(M)$, where $s_k := \#\{\sigma : dim\ \sigma = k\}$, as claimed. (Pict. below illustrates our construction.) ■ Let $V_0 := V_T$.

Next, constructions and proofs:



1. **homotopy of** V_0 **to** V_1 **,**

2. **then** $Ind(V_0) = Ind(V_1)$ **.**

Note. We'll use Riemanian metric on M , functions $\{f_j\}_{j\geq 1}$ from "Morse property is generic" (page 6) and $\Sigma_s(p) := \sum_{j\geq 1} s_j(\bigtriangledown f_j)(p)$ in a similar way: using PT Lemma we conclude for a.e. $s \in \mathbb{R}^N$, $\Phi_s :=$

$(1-t)V_0(p) + tV_1(p) + t(1-t)\cdot\Sigma_s(p)$ and map $F_s(p,t) := (p, \Phi_s(p,t))$

from $M \times [0.1] \to TM$, that $F_s \pitchfork 0_M$, where 0_M is the 0-section of TM .

Consequently: at $(p,t) \in \Phi_s^{-1}(0)$ maps $D\Phi_s(p,t)$ are onto $\Rightarrow$ $\Phi_s^{-1}(0)$

is a finite union of smooth arcs γ closed, or with ends in and tangents ξ_γ

transversal to $M \times \{t\}$, $t = 0, 1$ ($\Rightarrow$ $\xi_\gamma(p,t) \in \ker[D\Phi_s](p,t)$). Pick

continuous $T^\perp\gamma_{p,t} \not\ni \xi_\gamma(p,t)$, $(p,t) \in \gamma$, equal $T_pM \times \{0\}$ at $t = 0, 1$

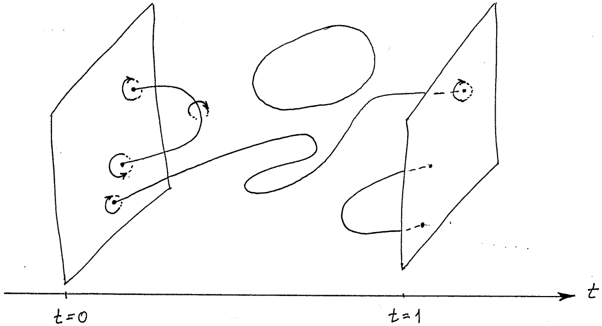
$\Rightarrow$ $\Lambda_{p,t} := D\Phi_s(p,t)_{T^\perp\gamma_{p,t}}$ are isomorphisms equal $DV_t(p)$ for $t = 0, 1$.

We next show 2 **for vector field** V_1 **not degenerate at all** $p \in V_1^{-1}(0)$ **.**

Moving along arc γ positive at γ's end continuous frame $\mathcal{F}_{p,t}$ in $T^{\perp}\gamma_{p,t}$, $(p,t) \in \gamma$, results in the oppositely or similarly oriented frame at the other end of γ depending on γ returning to the same value of t , i.e. 0 or 1 , or not. But orientation of continuous $\Psi_{p,t} := \Lambda_{p,t}(\mathcal{F}_{p,t})$ in TM is preserved.

Hence, due to the index being the sign of det of the tangent map (p. 9) it follows that the indexes at the ends of the 'returning' arcs cancel, while at the ends of the 'other' arcs equal $\Rightarrow$ $Ind(V_0) = Ind(V_1)$. ■

$Ind(V_0) = Ind(V_1)$ is as required. See Picture:



References

Lectures on Morse Homology by Augustin Banyaga and David Hurtubise

Poincare-Hopf Degree

proof:http://math.uchicago.edu/ amwright/PoincareHopf.pdf

Partition of Unity Theorem:

http://isites.harvard.edu/fs/docs/icb.topic134696.files/Partitions$_{o}f_{U}nity.pdf$

Morse Genericity:http://www.math.toronto.edu/mgualt/Morse