De Rham Theorem à la Whitney, Part 2.

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Towards $ker(\mathrm{Int}^\bullet)$ is acyclic. Extension of Forms Thm:

(a_k) Say $k \geq 0$, $s \geq 1$, σ an s-simplex and $\omega \in \Omega^k(U(\partial\sigma))$ is closed.

Assume $\int_{\partial\sigma} \omega = 0$ if $s = k+1$. Then exists $\tilde{\omega} \in \Omega^k(U(\sigma))$ closed and

s.th. $\tilde{\omega}_{|_{U(\partial\sigma)}} = \omega$ holds, perhaps upon shrinking $U(\partial\sigma)$.

(b_k) Say $k \geq 1$, $s \geq 1$, σ an s-simplex, $\omega \in \Omega^k(U(\sigma))$ closed, and

$\alpha \in \Omega^{k-1}(U(\partial\sigma))$, $U(\partial\sigma) \subset U(\sigma)$, s.th. $d\alpha = \omega_{|_{U(\partial\sigma)}}$. When $s = k$

assume $\int_\sigma \omega = \int_{\partial\sigma} \alpha$. Then exists $\tilde{\alpha} \in \Omega^{k-1}(U(\sigma))$ s.th. $\tilde{\alpha}_{|_{U(\partial\sigma)}} = \alpha$

and $d\tilde{\alpha} = \omega$, perhaps upon shrinking both $U(\partial\sigma) \subset U(\sigma)$.

Proof of $ker(\mathrm{Int}^\bullet)$ is acyclic, induction on $s \leq n$:

Say $L_s := \bigcup_i \sigma_i^s$ and $\omega \in ker(Int^k)$ is closed. Our plan is to construct

inductively nbds $U(L_s)$ of L_s and forms $\alpha_s \in \Omega^{k-1}(U(L_s))$ s.th.

$\alpha_{s|_{U(L_s)\cap U(L_{s-1})}} = \alpha_{s-1}$, $d\alpha_s = \omega_{|_{U(L_s)}}$ and $\mathrm{Int}^{k-1}(\alpha_{k-1}) = 0$. Then

$\alpha_n \in ker(\mathrm{Int}^{k-1})$ and $d\alpha_n = \omega$, proving that $ker(\mathrm{Int}^\bullet)$ is acyclic.

Choose disjoint, contractible nbds $U(\sigma_i^0)$. By Poincare Lemma exists

$\alpha_0' \in \Omega^0(U(\sigma_i^0))$ with $d\alpha_0' = \omega_{|_{U(\sigma_i^0)}}$. Set $\alpha_0 := \alpha_0'$ for $k > 1$ and

$\alpha_0 := \alpha_0' - \alpha_0'(\sigma_i^0)$ for $k = 1 \ \Rightarrow\ \mathrm{Int}^0(\alpha_0) = 0$, as required for $s = 0$.

Proof of $ker(\mathrm{Int}^\bullet)$ is acyclic, inductive step:

Given α_{s-1} , for each σ_i^s we now construct nbds $U(\sigma_i^s)$ s.th. overlaps of each two are subsets of $U(L_{s-1})$ and, also, forms $\alpha_s \in \Omega^{k-1}(U(\sigma_i^s))$ that coincide with α_{s-1} on overlaps. Inductive assumption includes $d\alpha_{s-1} = \omega_{|_{U(L_{s-1})}}$ and $\alpha_{s-1} \in ker(\mathrm{Int}^{k-1}(U(L_{s-1})))$ for $s = k$. Then (b_k) gives $\tilde{\alpha}_s^i \in \Omega^{k-1}(U(\sigma_i^s))$ s.th. $d\tilde{\alpha}_s^i = \omega_{|_{U(\sigma_i^s)}}$ and $\tilde{\alpha}^i_{s|_{U(\partial\sigma_i^s)}} = \alpha_{s-1}$. Shrink as in top 2 lines and glue $\tilde{\alpha}_s^i$ into $\tilde{\alpha}_s$ on $U(L_s) := \bigcup_i U(\sigma_i^s)$. We set $\alpha_s := \tilde{\alpha}_s$ for $s \neq k-1$ and $\alpha_s := \tilde{\alpha}_s - \Phi^{k-1}(\mathrm{Int}^{k-1}(\alpha_s))$ for $s = k-1$.

$\Phi^\bullet$ and $\mathrm{Int}^\bullet$ are homomorpisms of complexes and the former is the right inverse of the latter imply $d\alpha_{k-1} = \omega - \Phi^k(\mathrm{Int}^k(\omega)) = \omega$ on $U(L_s)$ and, also, that $\mathrm{Int}^{k-1}(\alpha_{k-1}) = \mathrm{Int}^{k-1}(\tilde{\alpha}_{k-1}) - \mathrm{Int}^{k-1}(\tilde{\alpha}_{k-1}) = 0$. ■

Proof of the Extension of Forms Theorem: by induction on k .

Plan: show (a_0) holds, then $(a_{k-1}) \Rightarrow (b_k)$ and, finally, $(b_k) \Rightarrow (a_k)$.

<u>(a_0):</u> Say $\omega \in \Omega^0(U(\partial\sigma))$ closed. Then ω is locally constant. Moreover, then $\omega \equiv c$ is constant since $\partial\sigma$ is connected when $s > 1$ and $0 = \int_{\partial\sigma} \omega := \omega(p_1) - \omega(p_0)$ when $\sigma = p_0 p_1$.

$\underline{(a_{k-1}) \Rightarrow (b_k)\ :}$ Say ω, α are as in (b_k) . Poincare Lemma provides

$\alpha' \in \Omega^{k-1}(U(\sigma))$ s.th. $d\alpha' = \omega$. Then $\beta := (\alpha - \alpha') \in \Omega^{k-1}(U(\partial\sigma))$ is

closed and when $s = k$ also $\int_{\partial\sigma} \beta = \int_{\partial\sigma} \alpha - \int_{\partial\sigma} \alpha' = \int_{\sigma} \omega - \int_{\sigma} \omega = 0$.

Applying (a_{k-1}) to β provides its closed extension $\tilde{\beta} \in \Omega^{k-1}(U(\sigma))$.

Then $\tilde{\alpha} := (\tilde{\beta} + \alpha') \in \Omega^{k-1}(U(\sigma))$ is as required in (b_k) due to the

constructions of α' , β and $\tilde{\beta}$ being closed.

$\underline{(b_k) \Rightarrow (a_k)\ :}$ Say $\sigma = p_0 \ldots p_s$ and ω are as in (a_k) , $k > 0$. Also,

$\sigma' := p_1 \ldots p_s$, $\mathcal{P}$ is the union of all faces of σ with p_0 as a vertex and

$U(\mathcal{P})$ is a contractible nbd s.th. $\mathcal{P} \subset U(\mathcal{P}) \subset U(\partial\sigma)$. Poincare Lemma gives $\alpha' \in \Omega^{k-1}(U(\mathcal{P}))$ s.th. $d\alpha' = \omega|_{U(\mathcal{P})}$; say nbd $U(\partial\sigma') \subset U(\mathcal{P})$. With $A := (\partial\sigma - \sigma') \in \Sigma_k$, $s = k+1 \Rightarrow \partial A = -\partial\sigma'$, $SuppA = \mathcal{P}$ and $\int_{\sigma'} \omega - \int_{\partial\sigma'} \alpha' = \int_{\sigma'} \omega + \int_A d\alpha' = \int_{\partial\sigma} \omega = 0$ by the assumptions on ω in (a_k) . Applying now (b_k) to simplex σ' and forms ω , α' provides $\tilde{\alpha}' \in \Omega^{k-1}(U(\sigma'))$ with $\tilde{\alpha}'|_{U(\partial\sigma')} = \alpha'$ and $d\tilde{\alpha}' = \omega|_{U(\sigma')}$. Shrink $U(\mathcal{P})$ so that $U(\mathcal{P}) \cap U(\sigma') \subset U(\partial\sigma')$, let $U(\partial\sigma) := U(\mathcal{P}) \cup U(\sigma')$ and set $\alpha \in \Omega^{k-1}(U(\partial\sigma))$ by $\alpha = \alpha'$ on $U(\mathcal{P})$ and $\alpha = \tilde{\alpha}'$ on $U(\sigma')$.

Extending, e.g. as 0 , smoothly by means of partition of unity, to form

$\alpha \in \Omega^{k-1}(U(\sigma))$ provides the required in (a_k) closed form $\tilde{\omega} := d\alpha$ since

$\tilde{\omega}_{|\partial\sigma} = d\alpha_{|\partial\sigma} = \omega$ due to the construction of forms α' and $\tilde{\alpha}'$. ∎

Application towards $\chi(M)$: