

# De Rham Theorem à la Whitney, Part 2.

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## Towards $\ker(\text{Int}^\bullet)$ is acyclic. Extension of Forms Thm:

( $a_k$ ) Say  $k \geq 0$ ,  $s \geq 1$ ,  $\sigma$  an  $s$ -simplex and  $\omega \in \Omega^k(U(\partial\sigma))$  is closed.

Assume  $\int_{\partial\sigma} \omega = 0$  if  $s = k + 1$ . Then exists  $\tilde{\omega} \in \Omega^k(U(\sigma))$  closed and s.th.  $\tilde{\omega}|_{U(\partial\sigma)} = \omega$  holds, perhaps upon shrinking  $U(\partial\sigma)$ .

( $b_k$ ) Say  $k \geq 1$ ,  $s \geq 1$ ,  $\sigma$  an  $s$ -simplex,  $\omega \in \Omega^k(U(\sigma))$  closed, and

$\alpha \in \Omega^{k-1}(U(\partial\sigma))$ ,  $U(\partial\sigma) \subset U(\sigma)$ , s.th.  $d\alpha = \omega|_{U(\partial\sigma)}$ . When  $s = k$

assume  $\int_\sigma \omega = \int_{\partial\sigma} \alpha$ . Then exists  $\tilde{\alpha} \in \Omega^{k-1}(U(\sigma))$  s.th.  $\tilde{\alpha}|_{U(\partial\sigma)} = \alpha$

and  $d\tilde{\alpha} = \omega$ , perhaps upon shrinking both  $U(\partial\sigma) \subset U(\sigma)$ .

## Proof of $\ker(\text{Int}^\bullet)$ is acyclic, induction on $s \leq n$ :

Say  $L_s := \bigcup_i \sigma_i^s$  and  $\omega \in \ker(\text{Int}^k)$  is closed. Our plan is to construct inductively nbds  $U(L_s)$  of  $L_s$  and forms  $\alpha_s \in \Omega^{k-1}(U(L_s))$  s.th.

$\alpha_s|_{U(L_s) \cap U(L_{s-1})} = \alpha_{s-1}$  ,  $d\alpha_s = \omega|_{U(L_s)}$  and  $\text{Int}^{k-1}(\alpha_{k-1}) = 0$  . Then  $\alpha_n \in \ker(\text{Int}^{k-1})$  and  $d\alpha_n = \omega$  , proving that  $\ker(\text{Int}^\bullet)$  is acyclic.

Choose disjoint, contractible nbds  $U(\sigma_i^0)$  . By Poincare Lemma exists

$\alpha'_0 \in \Omega^0(U(\sigma_i^0))$  with  $d\alpha'_0 = \omega|_{U(\sigma_i^0)}$  . Set  $\alpha_0 := \alpha'_0$  for  $k > 1$  and

$\alpha_0 := \alpha'_0 - \alpha'_0(\sigma_i^0)$  for  $k = 1 \Rightarrow \text{Int}^0(\alpha_0) = 0$  , as required for  $s = 0$  .

## Proof of $\ker(\text{Int}^\bullet)$ is acyclic, inductive step:

Given  $\alpha_{s-1}$ , for each  $\sigma_i^s$  we now construct nbds  $U(\sigma_i^s)$  s.th. overlaps of each two are subsets of  $U(L_{s-1})$  and, also, forms  $\alpha_s \in \Omega^{k-1}(U(\sigma_i^s))$

that coincide with  $\alpha_{s-1}$  on overlaps. Inductive assumption includes

$d\alpha_{s-1} = \omega|_{U(L_{s-1})}$  and  $\alpha_{s-1} \in \ker(\text{Int}^{k-1}(U(L_{s-1})))$  for  $s = k$ . Then  $(b_k)$

gives  $\tilde{\alpha}_s^i \in \Omega^{k-1}(U(\sigma_i^s))$  s.th.  $d\tilde{\alpha}_s^i = \omega|_{U(\sigma_i^s)}$  and  $\tilde{\alpha}_s^i|_{U(\partial\sigma_i^s)} = \alpha_{s-1}$ . Shrink

as in top 2 lines and glue  $\tilde{\alpha}_s^i$  into  $\tilde{\alpha}_s$  on  $U(L_s) := \bigcup_i U(\sigma_i^s)$ . We set

$\alpha_s := \tilde{\alpha}_s$  for  $s \neq k-1$  and  $\alpha_s := \tilde{\alpha}_s - \Phi^{k-1}(\text{Int}^{k-1}(\alpha_s))$  for  $s = k-1$ .

$\Phi^\bullet$  and  $\text{Int}^\bullet$  are homomorphisms of complexes and the former is the right inverse of the latter imply  $d\alpha_{k-1} = \omega - \Phi^k(\text{Int}^k(\omega)) = \omega$  on  $U(L_s)$  and, also, that  $\text{Int}^{k-1}(\alpha_{k-1}) = \text{Int}^{k-1}(\tilde{\alpha}_{k-1}) - \text{Int}^{k-1}(\tilde{\alpha}_{k-1}) = 0$ . ■

**Proof of the Extension of Forms Theorem:** by induction on  $k$ .

**Plan:** show  $(a_0)$  holds, then  $(a_{k-1}) \Rightarrow (b_k)$  and, finally,  $(b_k) \Rightarrow (a_k)$ .

$(a_0)$ : Say  $\omega \in \Omega^0(U(\partial\sigma))$  closed. Then  $\omega$  is locally constant. Moreover,

then  $\omega \equiv c$  is constant since  $\partial\sigma$  is connected when  $s > 1$  and

$$0 = \int_{\partial\sigma} \omega := \omega(p_1) - \omega(p_0) \quad \text{when } \sigma = p_0p_1 .$$

$(a_{k-1}) \Rightarrow (b_k)$  : Say  $\omega, \alpha$  are as in  $(b_k)$  . Poincare Lemma provides  $\alpha' \in \Omega^{k-1}(U(\sigma))$  s.th.  $d\alpha' = \omega$  . Then  $\beta := (\alpha - \alpha') \in \Omega^{k-1}(U(\partial\sigma))$  is closed and when  $s = k$  also  $\int_{\partial\sigma} \beta = \int_{\partial\sigma} \alpha - \int_{\partial\sigma} \alpha' = \int_{\sigma} \omega - \int_{\sigma} \omega = 0$  . Applying  $(a_{k-1})$  to  $\beta$  provides its closed extension  $\tilde{\beta} \in \Omega^{k-1}(U(\sigma))$  . Then  $\tilde{\alpha} := (\tilde{\beta} + \alpha') \in \Omega^{k-1}(U(\sigma))$  is as required in  $(b_k)$  due to the constructions of  $\alpha'$  ,  $\beta$  and  $\tilde{\beta}$  being closed.

$(b_k) \Rightarrow (a_k)$  : Say  $\sigma = p_0 \dots p_s$  and  $\omega$  are as in  $(a_k)$  ,  $k > 0$  . Also,  $\sigma' := p_1 \dots p_s$  ,  $\mathcal{P}$  is the union of all faces of  $\sigma$  with  $p_0$  as a vertex and

$U(\mathcal{P})$  is a contractible nbd s.th.  $\mathcal{P} \subset U(\mathcal{P}) \subset U(\partial\sigma)$ . Poincare Lemma gives  $\alpha' \in \Omega^{k-1}(U(\mathcal{P}))$  s.th.  $d\alpha' = \omega|_{U(\mathcal{P})}$ ; say nbd  $U(\partial\sigma') \subset U(\mathcal{P})$ .

With  $A := (\partial\sigma - \sigma') \in \Sigma_k$ ,  $s = k + 1 \Rightarrow \partial A = -\partial\sigma'$ ,  $\text{Supp} A = \mathcal{P}$

and  $\int_{\sigma'} \omega - \int_{\partial\sigma'} \alpha' = \int_{\sigma'} \omega + \int_A d\alpha' = \int_{\partial\sigma} \omega = 0$  by the assumptions on  $\omega$  in  $(a_k)$ . Applying now  $(b_k)$  to simplex  $\sigma'$  and forms  $\omega$ ,  $\alpha'$  provides

$\tilde{\alpha}' \in \Omega^{k-1}(U(\sigma'))$  with  $\tilde{\alpha}'|_{U(\partial\sigma')} = \alpha'$  and  $d\tilde{\alpha}' = \omega|_{U(\sigma')}$ . Shrink  $U(\mathcal{P})$

so that  $U(\mathcal{P}) \cap U(\sigma') \subset U(\partial\sigma')$ , let  $U(\partial\sigma) := U(\mathcal{P}) \cup U(\sigma')$  and

set  $\alpha \in \Omega^{k-1}(U(\partial\sigma))$  by  $\alpha = \alpha'$  on  $U(\mathcal{P})$  and  $\alpha = \tilde{\alpha}'$  on  $U(\sigma')$ .

Extending, e.g. as 0, smoothly by means of partition of unity, to form  $\alpha \in \Omega^{k-1}(U(\sigma))$  provides the required in  $(a_k)$  closed form  $\tilde{\omega} := d\alpha$  since  $\tilde{\omega}|_{\partial\sigma} = d\alpha|_{\partial\sigma} = \omega$  due to the construction of forms  $\alpha'$  and  $\tilde{\alpha}'$ . ■


Application towards  $\chi(M)$  :

$$v = 1 + 4 \cdot 2 + 8 + 1 = 18$$

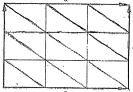
$$e = 4 \cdot 3 + 12 \cdot 4 = 60$$

$$t = 10 \cdot 4 = 40$$

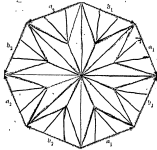
$$\chi = 18 - 60 + 40 = -2$$



$S^2; g=0$   
 $v=4, e=6, t=4$   
 $\chi(S^2) = 2 = 2 - 2g$



$T^2; g=1$   
 $v=9, e=27, t=18$   
 $\chi(T^2) = 0 = 2 - 2g$



$C(2\text{-handles}); g=2$   
 $v=19, e=54, t=24$   
 $\chi(C) = -2 = 2 - 2g$

Home Assignment: for  $g > 2$  it follows that  $\chi(C) = 2 - 2g$ .