Central Limit Theorem using Characteristic functions

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Introduction—study a random variable

Let $\Omega$ with measure $m$, $m(\Omega) = 1$ and $\mathcal{F}(\Omega)$ measurable functions.

To random variables $X \in \mathcal{F}(\Omega)$ we associate distribution functions

$$F(x) := P(X < x) := m(\xi \in \Omega : X(\xi) < x)$$

and $f(x) = F'(x)$ is the probability density function (shortly pdf). We assume exist finite:

Expected value of $X$ (mean value) $\mu = E(X) = \int_{\mathbb{R}} x dF(x) = \int_{\Omega} X(\xi) dm$

Variance and the standard deviation $\sigma^2 = V(X) = \int_{\mathbb{R}} (x - \mu)^2 dF(x)$.
Convention: \( P (...) := m (...) \) and for \( \{ A_j \subset \mathbb{R} \}_{1 \leq j \leq n} \) and 

\[ \{ X_j \in \mathcal{F}(\Omega) \}_j \text{ set } X_1 \in A_1, \cdots, X_n \in A_n := \{ \xi \in \Omega : X_j(\xi) \in A_j, \forall j \} \]

For our \( X_j \) distribution function, expected value and variance are the same and \( \{ X_j \}_j \) are independent, identically distributed (shortly iid), i.e. \( P (X_1 \in A_1, \cdots, X_n \in A_n) = \prod_{j=1}^{n} P (X_j \in A_j) \).

For iid \( \{ X_j \}_j \) set \( S_n := \frac{\sum_{1}^{n} X_i}{n} \) and also

\[
E (S_n) = E \left( \frac{\sum_{1}^{n} X_i}{n} \right) = \frac{\sum_{1}^{n} E(X_i)}{n} = \frac{n \cdot \mu}{n} = \mu
\]

\[
V (S_n) = V \left( \frac{\sum_{1}^{n} X_i}{n} \right) = \frac{V(\sum_{1}^{n} X_i)}{n^2} = \frac{n \cdot V(X)}{n^2} = \frac{V(X)}{n} = \frac{\sigma^2}{n}.
\]
Law of Large Numb: \( \lim_{n \to \infty} P \left( |S_n - \mu| > \epsilon \right) = 0, \forall \epsilon > 0 \)

Proof: \[ \frac{\sigma^2}{n} = V(S_n) = \int_R (x - \mu)^2 dF(x) \geq \int_{|x-\mu|>\epsilon} (x - \mu)^2 dF(x) \]

\[ \geq \epsilon^2 \cdot P \left( |S_n - \mu| > \epsilon \right) \Rightarrow P \left( |S_n - \mu| > \epsilon \right) \leq \delta \text{ for } n > \frac{\sigma^2}{\epsilon^2 \cdot \delta} . \]

Def: \( X_n \overset{d}{\to} X \), i.e. converge in the sense of distributions means

\[ \forall \text{ bounded and continuous function } f : \int_R f dF_n \to \int_R f dF . \]

Central Limit Theorem (shortly CLT): \( \frac{(S_n-\mu)\sqrt{n}}{\sigma} \overset{d}{\to} N(0,1) \), where

\[ S_n = \frac{\sum_{i=1}^{n} X_i}{n} \text{ and } N(0,1) \text{ is the rv with pdf } \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \text{ of Gauss distribution} \]
Next, note that \( N(0,1) \) has expected value \( \int_{R} x dF(x) = 0 \) and variance \( \int_{R} x^2 dF(x) = 1 \). Also, \( \{X_j\}_j \) being iid’s of course (page 3)

\[
E(S_n) = \mu, \quad V(S_n) = \frac{\sigma^2}{n}.
\]

To prove the theorem we’ll use the characteristic functions \( \varphi(t) = E(e^{itX}) = \int_{R} e^{itx} dF(x) \), shortly cfs

**Note:** rvs always admit cfs; \( \varphi'(0) = i\mu \) and \( \mu = 0 \) \( \Rightarrow \varphi''(0) = -\sigma^2 \).

Also \( \varphi(0) = E(1) = 1, \quad \mid \varphi(t) \mid = \mid \int_{R} e^{itx} dF(x) \mid \leq \int_{R} \mid e^{itx} \mid dF(x) = 1. \)

**Fact1:** \( F(b) - F(a) = \frac{1}{2\pi} \lim_{x \to \infty} \int_{-x}^{x} e^{-ita}e^{-itb} \frac{1}{it} \varphi(t) dt \).

**Easy if** \( \exists F'(x) : f(x) = \frac{1}{2\pi} \int_{R} e^{-itx} \varphi(t) dt. \)
Properties of characteristic function

\[ \varphi_{X_1+X_2}(t) = E(e^{it(X_1+X_2)}) = E(e^{itX_1}) \cdot E(e^{itX_2}) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t) \]

\[ \varphi_{aX+b}(t) = E(e^{it(aX+b)}) = e^{itb} \cdot E(e^{i(at)X}) = e^{itb} \cdot \varphi_X(at) \]

and also the uniform continuity of cf with \( \mu = 0 \) and \( \sigma = 1 \) :

\[ |\varphi(t+h) - \varphi(t)| = |E(e^{i(t+h)X} - e^{iX})| \leq E(|e^{ihX} - 1|) \to 0 \]

Characteristic function for Gauss distribution is \( e^{-\frac{t^2}{2}} \), page 14.
Convergence of F implies convergence of cfs

Proposition: \( X_n \xrightarrow{d} X \iff \varphi_n(x) \rightarrow \varphi(x) \quad \forall x \in \mathbb{R} \).

Proof. "\( \Rightarrow \)": \( e^{itx} \) is bounded and continuous and \( X_n \xrightarrow{d} X \) imply \( \int_{\mathbb{R}} e^{itx} dF_n \rightarrow \int_{\mathbb{R}} e^{itx} dF \). To show "\( \Leftarrow \)" (see page 12) we need to prove first a so called ‘tightness’ of our rvs.

Tightness of a family of Random Variables.

Def: a family of rvs \( X_n \) is tight when

\[ \forall \epsilon > 0 \ \exists M \text{ such that } P(|X_n| > M) < \epsilon \quad \text{for all } n. \]
Claim: convergence of cfs implies tightness of rvs.

Proof of 1st step: we show that for any distribution $X := X_n$,

$$\forall \epsilon > 0 \exists M, \ P(|X| > M) < \frac{\epsilon}{2}.$$

Indeed, every cf has a value of 1 at 0 (page 5) and is continuous

$$\Rightarrow \forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall |t| < \delta, \ |1 - \varphi(t)| < \frac{\epsilon}{4}$$

$$\Rightarrow \int_{-\delta}^{\delta} |1 - \varphi(t)| \ dt < 2\delta \cdot \frac{\epsilon}{4} = \frac{\epsilon \cdot \delta}{2}$$

$$\Rightarrow \delta^{-1} \int_{-\delta}^{\delta} |1 - \varphi(t)| \ dt < \frac{\epsilon}{2}.$$  On the other hand, for some large $M$

$$\delta^{-1} \int_{-\delta}^{\delta} |1 - \varphi(t)| \ dt \text{ is an upper bound on } P(|X| \geq M).$$
\[ \delta^{-1} \int_{-\delta}^{\delta} (1 - \varphi(t)) \, dt = \delta^{-1} \int_{-\delta}^{\delta} (1 - E(e^{itX})) \, dt = \delta^{-1} \left( 2\delta - \int_{\mathbb{R}} \left( \frac{\sin(\delta x)}{x} - \frac{\sin(-\delta x)}{x} \right) \, dF(x) \right) = 2 \left( 1 - \int_{\mathbb{R}} \frac{\sin(\delta x)}{\delta x} \, dF(x) \right). \]

Replacing 1 by \( \int_{\mathbb{R}} 1 \, dF(x) \), we have \( 2 \left( 1 - \int_{\mathbb{R}} \frac{\sin(\delta x)}{\delta x} \, dF(x) \right) = 2 \int_{\mathbb{R}} \left( 1 - \frac{\sin(\delta x)}{\delta x} \right) \, dF(x). \)

\[ 2 \int_{\mathbb{R}} \left( 1 - \frac{\sin(\delta x)}{\delta x} \right) \, dF(x) \geq 2 \int_{|x| \geq \frac{1}{\delta}} \left( 1 - \frac{\sin(\delta x)}{\delta x} \right) \, dF(x) \geq \int_{|x| \geq \frac{2}{\delta}} 1 \, dF(x) = P(|X| \geq \frac{2}{\delta}) \quad \Rightarrow \quad \delta^{-1} \int_{-\delta}^{\delta} |1 - \varphi(t)| \, dt \geq P(|X| \geq \frac{2}{\delta}). \]

Together with above \( \frac{\epsilon}{2} > \delta^{-1} \int_{-\delta}^{\delta} |1 - \varphi(t)| \, dt \geq P(|X| \geq \frac{2}{\delta}). \) □
Step 2: convergence of cfs implies tightness in its rvs

\[ \phi_n(x) \to \phi(x) \] means \( \forall \epsilon > 0 \), \( x \in \mathbb{R} \) \( \exists \) natural number \( N \) s. th.

\[ \forall \ n > N \ holds \ |\phi_n(x) - \phi(x)| < \frac{\epsilon}{4} \Rightarrow \forall \ \epsilon, \ \delta \ \exists \ N \ such \ that \]

\[ \forall \ n \geq N \ we \ have \ \delta^{-1} \int_{-\delta}^{\delta} |\phi_n(t) - \phi(t)| \ dt < \frac{\epsilon}{2} \ (\text{fact from analysis}). \]

Also, (page 9) we may choose \( \delta \) to satisfy \( \delta^{-1} \int_{-\delta}^{\delta} |1 - \phi(t)| \ dt < \frac{\epsilon}{2} \)

\[ \forall \ n \geq N \ we \ have \ \mathbb{P}(|X_n| \geq \frac{2}{\delta}) \leq \delta^{-1} \int_{-\delta}^{\delta} |1 - \phi_n(t)| \ dt \]
\[
\leq \delta^{-1} \left( \int_{-\delta}^{\delta} |1 - \varphi(t)| \, dt + \int_{-\delta}^{\delta} |\varphi_n(t) - \varphi(t)| \, dt \right) < \epsilon.
\]

Also, for \( n \) smaller than \( N \) \( \exists \delta_n \) such that

\[
P \left( |X_n| \geq \frac{2}{\delta_n} \right) \leq \delta^{-1}_n \int_{-\delta_n}^{\delta_n} |1 - \varphi_n(t)| \, dt < \epsilon
\]

choose \( \delta_{min} := \min \{ \delta_1, \delta_2, \cdots, \delta_n, \delta \} \).

we have then that \( P \left( |X_n| \geq \frac{2}{\delta_{min}} \right) < \epsilon \) for any \( n \)

\( \Rightarrow \) rvs with convergent cfs are tight, the claim is proved. \( \square \)
Proof of $\varphi_n(x) \to \varphi(x) \ \forall \ x$ implies $X_n \xrightarrow{d} X$ using

**Fact 2.** Tightness of rvs implies compactness in the sense of convergence of distributions ("Prokhorov’s Theorem").

**Proof of "$\Leftarrow$" from page 7:** Pick any convergent, say to $F_1$, subsequence $\{F_{1n}\}_n$ of distributions. Say $\{\varphi_{1n}\}_n$ are their cfs.

$\varphi_n(x) \to \varphi(x) \ \forall \ x$ implies convergence of all $\{\varphi_{1n}\}_n$ to the same $\varphi$ and proved "$\Rightarrow$" on page 7 implies that $\varphi$ is the cf for any $F_1$

$\Rightarrow$ exists unique $F_1 =: F$ and, using **Fact 2.**, $\Rightarrow$ $X_n \xrightarrow{d} X$, i.e.

$$X_n \xrightarrow{d} X \Leftrightarrow \varphi_n(x) \to \varphi(x) \ \forall \ x$$

is proved. $\Box$
Conclusion of the proof of Central Limit Theorem

For a series of iid $X_i$, let $Y_n = \frac{\sum_{1}^{n} X_i - n\mu}{\sigma \sqrt{n}}$

$\varphi_Y(t) = \varphi_{\frac{\sum_{1}^{n} X_i - n\mu}{\sigma \sqrt{n}}}(t) = \varphi_{\frac{\sum_{1}^{n} X_i - n\mu}{\sigma \sqrt{n}}}(\frac{t}{\sigma \sqrt{n}}) =: \varphi^n(\frac{t}{\sigma \sqrt{n}})$. Let $s := \frac{t}{\sigma \sqrt{n}}$, as $n \to \infty s \to 0$. Recall: $\varphi'(0) = i\mu = 0$, $\varphi''(0) = -\sigma^2$, see page 5.

From Taylor expansion: $\varphi(0) + s \cdot \varphi'(0) + \frac{s^2}{2} \cdot \varphi''(0) - \varphi(s) = o(s^2) \Rightarrow$

$$\lim_{n \to \infty} \{ \varphi_Y(t) = \left(1 - \frac{\sigma^2 + o(1)}{2} \cdot \left(\frac{t}{\sigma \sqrt{n}}\right)^2\right)^n \} = \lim_{n \to \infty} \left(1 - \frac{\sigma^2}{2} \left(\frac{t}{\sigma \sqrt{n}}\right)^2\right)^n$$

$= e^{-\frac{t^2}{2}} \Rightarrow$ the limit of the cfs is the cf of a Gauss distribution

$\Rightarrow \frac{(S_n - \mu)\sqrt{n}}{\sigma} = Y_n \xrightarrow{d} N(0, 1)$, as required. □
Appendix. cf of Normal distribution, calculation:

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \]

\[ \varphi(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{itx} \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + itx - \frac{1}{2}(it)^2 + \frac{1}{2}(it)^2} \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + itx - \frac{1}{2}(it)^2} \, dx \]

\[ = e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\text{i}t)^2} \, dx . \]

\[ y = x - \text{i}t \Rightarrow \frac{dy}{dx} = 1 \Rightarrow \]

\[ \varphi(t) = e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy = e^{-\frac{t^2}{2}} . \]
Abbreviations

rv : random variable

rvs : random variables

pdf : probability density function

iid : independent, identical distributed rvs

cf : characteristic function

cfs : characteristic functions