Overview of First-Order Logic

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MAT 477

January 2014

Review of first-order logic

Def: The logical symbols are $\forall$, $\exists$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, (,),$=$, and the variable symbols $\mathsf{v}_1, \mathsf{v}_2, \ldots$

Def: Every non-logical symbol is either

- a constant symbol;
- an n-ary relation symbol for some $n \in \mathbb{N}$; or
- an n-ary function symbol for some $n \in \mathbb{N}$.

Def: A language $\mathcal{L}$ is a collection of symbols including all logical symbols and some non-logical symbols. Typically, we denote a language by its non-logical symbols. For instance, the language of arithmetic $\mathcal{L}_A$ is given by the constant symbol 0, the unary function symbol S [successor], and the binary function symbols $+$ and $\times$.

Def: Any finite sequence of symbols in the language $\mathcal{L}$ is called a $\mathcal{L}$-expression.

Def: We define $\mathcal{L}$-terms recursively as follows:

- Any constant symbol or variable symbol in $\mathcal{L}$ is a term;
- If f is an n-ary function and $t_1, \ldots, t_n$ are terms, then

  $f(t_1, \ldots, t_n)$ is a term.

e.g. $+(S(0), 0)$ is a $\mathcal{L}_A$-term. However, we allow ourself notational conveniences. For instance, we may express $+(S(0), 0)$ as $(S0) + 0$ so long as its understood what it represents. Likewise, we may informally use letters to represent variables, i.e. x instead of v_4.

Def: We define $\mathcal{L}$-formulas as follows:

- If t_1 and t_2 are $\mathcal{L}$-terms, then $\mathsf{t}_1 = \mathsf{t}_2$ is a formula;
- If R is an n-ary relation symbol in $\mathcal{L}$, and $\mathsf{t}_1, \ldots, \mathsf{t}_n$ are terms,

  then $\mathsf{R}(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ is a formula;
- If φ is a formula, then $\neg\varphi$ is a formula;
- φ and ψ formulas $\Rightarrow (\varphi \wedge \psi)$ is a formula. Similarly for $\vee$, $\rightarrow$, $\leftrightarrow$;
- φ formula, x variable symbol $\Rightarrow \forall \mathsf{x}\ \varphi$ and $\exists \mathsf{x}\ \varphi$ are formulas.

Note: Formulas can't talk about arbitrary sets, or arbitrary formulas.

Notation: Again, we allow ourselves some informal notational conveniences. For instance, we may write the formula $\forall v_1(v_1 > 0 \to \neg v_1 = 0)$ as $(\forall x > 0)(x \neq 0)$.

Def: A variable is called free in the formula φ if it occurs in φ not under a quantifier, i.e. v_1 is *free* in $v_1 = v_1$ and in $(\forall v_1\ v_1 = v_1) \wedge v_1 = v_1$. If $v_{i_1}, \ldots, v_{i_k}$ are free in φ, we may denote φ by $\varphi(v_{i_1}, \ldots, v_{i_k})$. Then, $\varphi(t, v_{i_2}, \ldots, v_{i_k})$ is the formula obtained by replacing all free occurences of v_{i_1} by term t. A *sentence* is a formula with no free variables.

Def: A *theory* T in the language $\mathcal{L}$ is a set of $\mathcal{L}$-sentences called *axioms*.

Peano Arithmetic is a theory in $\mathcal{L}_A$ including the following axioms:

1. $\forall x\ 0 \neq Sx$

2. $\forall x \forall y (Sx = Sy \rightarrow x = y)$

3. $\forall x (x + 0 = x)$

4. $\forall x \forall y (x + Sy = S(x + y))$

5. $\forall x (x \times 0 = 0)$

6. $\forall x \forall y (x \times Sy = (x \times y) + x)$

7. For every formula $\varphi(x)$, $\{\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx))\} \rightarrow \forall x \varphi(x)$

Note that Axiom 7 is not actually a single axiom but rather countably many distinct axioms, one for each formula φ.

Logical Axioms

Regardless of the theory, we always include the following axioms implicitly.

For any $\mathcal{L}$-formulas φ, ψ, ζ:

1. $\varphi \to (\psi \to \varphi)$

2. $(\varphi \to (\psi \to \zeta)) \to ((\varphi \to \psi) \to (\varphi \to \zeta))$

3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

are axioms.

Model: So far, we have discussed notation. We wish to attach meaning to notation. When we interpret $\forall$x, it is necessary to restrict ourselves to some domain. We also need an interpretation of each constant, function, and relation in $\mathcal{L}$. Formally,

Def: An $\mathcal{L}$-model is a set U, called the universe, together with:

- For each constant symbol c $\in \mathcal{L}$, some $c \in U$
- For each function symbol $\mathrm{f}^{\mathrm{k}} \in \mathcal{L}$, some function $f : U^k \to U$
- For each relation symbol $\mathrm{R}^{\mathrm{k}} \in \mathcal{L}$, some k-place relation R on U

Def: The *term-function* $T_{\mathsf{t}} : U^{\omega} \to U$ of the term t is given by:

- if t is a variable symbol $\mathsf{x_i}$,then $T_{\mathsf{t}}(a_1, a_2, \ldots) = a_i$;

- if t is a constant symbol c, then $T_{\mathsf{t}}(a_1, a_2, \ldots) = c$;

- if t is $\mathsf{f}(\mathsf{t}_1, \ldots, \mathsf{t}_\mathsf{k})$, then

$$T_{\mathsf{t}}(a_1, a_2, \ldots) = f(T_{\mathsf{t}_1}(a_1, \ldots,), T_{\mathsf{t}_2}(a_1, \ldots,), \ldots)$$

Def: The *truth-function* $F_\varphi : U^\omega \to \{0, 1\}$ of the formula φ is given by:

- If φ is $\mathsf{t}_1 = \mathsf{t}_2$, then $F_\varphi(a_1, \dots) = 1$ if $T_{\mathsf{t}_1}(a_1, \dots) = T_{\mathsf{t}_2}(a_1, \dots)$;

$F_\varphi(a_1, \dots) = 0$ otherwise;

- If φ is $\mathsf{R}(\mathsf{t}_1, \dots, \mathsf{t}_\mathsf{k})$, then $F_\varphi(a_1, \dots) = 1$ if it holds that

$R(T_{\mathsf{t}_1}(a_1, \dots), \dots, T_{\mathsf{t}_\mathsf{k}}(a_1, \dots))$; $F_\varphi(a_1, \dots) = 0$ otherwise;

- If φ is $\psi \wedge \zeta$, then $F_\varphi(a_1, \dots) = 1$ if $F_\psi(a_1, \dots) = 1$ and

$F_\zeta(a_1, \dots) = 1$; $F_\psi(a_1, \dots) = 0$ otherwise;

    - Defined similarly for $\psi \vee \zeta$, $\psi \to \zeta$, $\psi \leftrightarrow \zeta$;

- If φ is $\neg\psi$, then $F_\varphi(a_1, \dots) = 1$ if $F_\psi(a_1, \dots) = 0$;

- If φ is $\forall x_i\ \psi$, then $F_\varphi(a_1, \dots) = 1$ if for all $a_i' \in U$,

$F_\psi(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots) = 1$; $F_\varphi(a_1, \dots) = 0$ otherwise;

- If φ is $\exists x_i\ \psi$, $F_\varphi(a_1, \dots) = 1$ if there exists $a_i' \in U$

s.th. $F_\psi(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots) = 1$; $F_\varphi(a_1, \dots) = 0$ otherwise.

Def: Suppose φ is a sentence. Then its truth function F_φ is constant.

Say $\mathcal{M}$ *models* φ (write $\mathcal{M} \models \varphi$) if $F_\varphi = 1$.

The Standard Model: Herein, we are primarily concerned with the language $\mathcal{L}_A$ and its *standard model*. This model has $\mathbb{N}$ as its universe with 0 interpreted as the additive identity; S interpreted as the successor function; $+$ interpreted as addition; and $\times$ interpreted as multiplication.

We'll say the $\mathcal{L}_A$ sentence φ is *true* if it is modeled by the standard model.

Rule of Deduction: From φ and $\varphi \to \psi$, we can conclude ψ.

Provability: A *proof* of an $\mathcal{L}$-sentence φ in an $\mathcal{L}$-theory T is a finite sequence of $\mathcal{L}$-formulas ending with φ such that each formula is either an axiom or follows by the rule of deduction from some earlier formulas in the sequence. If a proof of φ exists, write $T \vdash \varphi$ and say φ is a theorem.

Fact (Soundness): Our rule of deduction is truth-preserving. Hence, if the axioms of T are modeled by $\mathcal{M}$, then $\mathcal{M}$ models every theorem of T.

Note: For every $\mathcal{L}_A$-formula φ, either φ or $\neg\varphi$ is true. However, we'll see that both φ and $\neg\varphi$ can be unprovable. Write $T \nvdash \varphi$ for φ unprovable.

A last note on notation: It is useful to borrow the notation of formal languages for our own 'mathematicians' language. In general, context will dictate which is which. However, herein we'll use italics (i.e. $\exists y\ y = S\mathrm{o}$) for our 'mathematician' language and non-italics (i.e. $\exists \mathrm{y}\ \mathrm{y} = \mathrm{S0}$) for our formal language. On the other hand, $\mathcal{L}_A$ has no numeral symbols except 0 so we'll use $\bar{\mathrm{n}}$ to denote $\mathrm{S} \ldots \mathrm{S0}$ (n times).