Chow's Theorem

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Definitions

$X \subset U$, open in $\mathbb{C}^n$, is **analytic** if it is closed and locally the set

of zeros of analytic functions $f_1, \ldots, f_k$.

$Reg(X)$=$\{x \in X : X$ is a manifold around $x\}$, and

$Sing(X) = X \setminus Reg(X)$.

$X \subset U$, is ***-analytic** if : $X = X^{(r)} \cup X^{(r-1)} \cup \cdots \cup X^{(0)}$

with $X^{(i)}$ a complex submanifold of U of dimension i and

$\overline{X^{(i)}} \subset X^{(i)} \cup \cdots \cup X^{(0)}$.

Preliminary Facts:

Thm 1. X analytic $\Rightarrow$ $Sing(X)$ analytic and

$dim(Sing(X)) < dim(X)$.

Cor. X analytic $\Rightarrow$ X is *-analytic.

Main Results

Thm 2. X is *-analytic $\Rightarrow$ X is analytic.

Cor. $X \subset \mathbb{P}^n$ is *-analytic $\Rightarrow$ X is algebraic.

Proof. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the canonical map. Then

$Z = \pi^{-1}(X) \cup \{0\}$ is *-analytic $\Rightarrow$ Z is analytic. Write near 0

$Z = V(f_1, \ldots, f_k)$. So $f_i = \sum_{r \geq 1} \sum_{|\alpha|=r} c_\alpha^{(i)} x^\alpha = \sum_r f_{i,r} = 0$

$\Rightarrow f_i(\lambda x) = \sum_r \lambda^r f_{i,r}(x) = 0, \forall |\lambda| < 1 \Rightarrow X = V(\ldots, f_{i,r}, \ldots)$. $\square$

Note: From now on all U's and V's are open in . . . .

Proof of Thm 2 via repeated

Inductive Step: Let $X = X^{(r)} \cup X' \subset U \subset \mathbb{C}^n$ s.th. $X^{(r)}$ is an r-manifold (r-dimensional manifold); $\overline{X^{(r)}} \subset X$; X' analytic, *-analytic and $dim(X') < r \Rightarrow \overline{X^{(r)}}$ is analytic.

Proposition A. $p : \mathbb{C}^{n+k} \to \mathbb{C}^n$ linear projection. $X \subset U \subset \mathbb{C}^{n+k}$ analytic, $V \subset \mathbb{C}^n$ with $p|_X : X \to V$ proper $\Rightarrow p(X)$ is analytic in V and $p|_X$ is finite-to-one.

Proof. Reduce to k=1. Factor p :

$$\mathbb{C}^{n+k} \xrightarrow{p_1} \mathbb{C}^{n+k-1} \xrightarrow{p_2} \mathbb{C}^n$$

Then $p|_X$ proper $\Rightarrow p_1|_X : X \to U_1 := p_1(U)$ is proper

$\Rightarrow X_1 = p_1(X)$ is analytic in U_1 ...

For k=1 : Let $y \in V$ then $X \cap p^{-1}(y)$ compact and analytic in

$U \cap p^{-1}(y) \Rightarrow X \cap p^{-1}(y) = \{x_1, \ldots, x_l\} \Rightarrow \exists U_1, \ldots, U_k$ open,

disjoint, with $x_j \in U_j$ and $W \subset V$ open s.th.

$X \cap p^{-1}(W) \subset U_1 \cup \cdots \cup U_k$.

It suffices to show that $p(X \cap p^{-1}(W) \cap U_i)$ is analytic. We may

assume WLOG that $y = 0$ and $p^{-1}(0) = \{0\}$.

Let $X = V(f_1, \ldots, f_k)$ near 0. By *Weierstrass Prep. Thm* we

may assume : $f_1 = z_{n+1}^d + a_1 z_{n+1}^{d-1} + \cdots + a_d$,

$f_i = b_{1,i} z_{n+1}^{d-1} + \cdots + b_{d,i}$, for $i \geq 2$.

With $a_j, b_{j,i} \in \mathbb{C}\{z_1, \ldots, z_n\}$, $a_j(0) = 0$.

$Res(f_1, t_2 f_2 + \cdots + t_k f_k) =: \sum_{|\alpha|=d} t^\alpha R_\alpha$ *with* $R_\alpha \in \mathbb{C}\{z_1, \ldots, z_n\}$.

Claim. $p(X) = V(\{R_\alpha\})$.

$(z_1, \ldots, z_n) \in p(X) \Rightarrow \sum_{|\alpha|=d} t^\alpha R_\alpha(z_1, \ldots, z_n) = 0 , \forall t \in \mathbb{C}^{k-1}$

$\Rightarrow R_\alpha(z_1, \ldots, z_n) = 0 , \forall \alpha$.

$(z_1, \ldots, z_n) \in V(\{R_\alpha\}) \Rightarrow W_1 \cup \cdots \cup W_d = \mathbb{C}^{k-1}$, where

$W_j := \{t \in \mathbb{C}^{k-1}, \sum t_i f_i(z_1, \ldots, z_n, A_j) = 0\}$, provided $\{A_j\}$ is a

root of $f_1(z_1, \ldots, z_n, z) \Rightarrow \exists j_0$, $W_{j_0} = \mathbb{C}^{k-1}$

$\Rightarrow (z_1, \ldots, z_n, A_{j_0}) \in X$. $\square$

Our case: $X_0 \cup X_1 \subset U \subset \mathbb{C}^n$, X_1 analytic in U and X_0 analytic

in $U \setminus X_1$. We may assume $X_0 \cup X_1 \neq U$ and $0 \in X_1$.

Lemma We may shrink U so that $\exists L \subset \mathbb{C}^n$ a line with $0 \in L$ s.th.

(i) $X_1 \cap L = \{0\}$.

(ii) $X_0 \cap L$ is a set of discrete points with only possible limit point 0 .

Cor 1. $(X_0 \cup X_1) \cap L$ is compact

Cor 2. We may shrink U so that the projection $p|_{X_0 \cup X_1}$ along L is proper (*Mumford's Lemma*).

Proof.

(i) $X_1 \cap L$ is analytic in $U \cap L$, so 0 is isolated in $U \cap L$.

Remains to shrink U.

(ii) $X_0 \cap L$ is analytic in $(U \setminus X_1) \cap L$, so similarly it suffices to shrink U.

Proposition A + Induction $\Rightarrow$ we can find a projection

$p : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ s.th. $V := p(X_0 \cup X_1)$ is an open nbhd of 0,

$p|_{X_0 \cup X_1}$ is proper and $p|_{X_0 \setminus p^{-1}(p(X_1))}$ is finite-to-one.

Then $m = dim(X_0)$ using Sard's Lemma.

In notation of inductive step (page 4):

$X = X^{(r)} \cup X'$ $(m = r)$, and $Y := p(X') \subset V := p(X)$.

$$\begin{array}{ccccc} X^{(r)} \setminus p^{-1}(Y) & \subset & X & \supset & X' \\ q \downarrow & & \downarrow p & & \downarrow \\ V \setminus Y & \subset & V & \supset & p(X') = Y \end{array}$$

Note that $V \setminus Y$ is open and dense in V and that if V is a ball

then it is connected (since $dim_{\mathbb{C}}(Y) < dim_{\mathbb{C}}(V)$).

Similarly $X^{(r)} \setminus p^{-1}(Y)$ is open and dense in $X^{(r)}$.

$q := p|_{X^{(r)}\setminus p^{-1}(Y)}$ is proper and finite-to-one, between two

r-manifold.

Let J be the jacobian of q. Then $A = J^{-1}(0)$ is a closed analytic

subset of $X^{(r)} \setminus p^{-1}(Y)$

$\Rightarrow B := q(A)$ is an analyitic subset of $V \setminus Y$, by proposition A.

By Sard's thm, $\tilde{V} := V \setminus (B \cup Y)$ is open, dense in V. Also $\tilde{V}$ is

connected ($dim_{\mathbb{C}}(B \cup Y) < dim_{\mathbb{C}}(V)$).

Similarly $\tilde{X^{(r)}} = X^{(r)} \setminus p^{-1}(B \cup Y)$ is open and dense in $X^{(r)}$.

So $s := q|_{\tilde{X^{(r)}}}$ is a proper, finite-to-one local diffeomorphism

onto a connected $\tilde{V} \subset \mathbb{C}^n$.

Definition of the functions:

(i) d the number of sheets of the covering s ;

(ii) $\mathcal{L}$ a linear function on $\mathbb{C}^n$;

Let σ_i be the i-th symmetric function, so that for

$\lambda_1, \ldots, \lambda_d \in \mathbb{C}$;

$\prod_i (z - \lambda_i) = z^d + \sum_i \sigma_i(\lambda_1, \ldots, \lambda_d) z^{d-i}$.

(iii) for $y \in \tilde{V}$, let $s^{-1}(y) = \{x_1, \ldots, x_d\}$ and define

$a_i(y) := \sigma_i(\mathcal{L}(x_1), \ldots, \mathcal{L}(x_d))$;

(iv) for $y \in V$ $\exists$ a nbhd W s.th. a_i is bounded on $\tilde{V} \cap W$

$\Rightarrow$ we may extend a_i as $\mathbb{C}$-anlytic functions on V,by

Riemann Extension Thm.

(v) $F_{\mathcal{L}}(x) := \mathcal{L}(x)^d + a_1(p(x))\mathcal{L}(x)^{d-1} + \cdots + a_d(p(x))$ on

$p^{-1}(V)$.

End of Proof of Thm 2.

Step 1. $F_{\mathcal{L}} \equiv 0$ on $\tilde{X^{(r)}} \Rightarrow F_{\mathcal{L}} \equiv 0$ on $\overline{X^{(r)}} \Rightarrow \overline{X^{(r)}} \subset \cap_{\mathcal{L}} V(F_{\mathcal{L}})$.

Step 2. For $x \in p^{-1}(V) \setminus \overline{X^{(r)}} \subset \mathbb{C}^n$, let $y = p(x)$, and $y_k \in \tilde{V}$

s.th. $\lim y_k = y$ ($\tilde{V}$ dense in V).

Let $s^{-1}(y_k) = \{x_k^1, \ldots, x_k^d\}$. Since $p : \overline{X^{(r)}} \to V$ is proper we

may assume, by choosing a subsequence, that $x_k^j \to x^j \in \overline{X^{(r)}}$.

Since $x \notin \overline{X^{(r)}}$, $\exists \mathcal{L} \in (\mathbb{C}^n)^*$ s.th. $\forall j$ we have$\mathcal{L}(x) \neq \mathcal{L}(x^j)$

$\Rightarrow F_{\mathcal{L}}(x) \neq 0 \Rightarrow \overline{X^{(r)}} = \cap_{\mathcal{L}} V(F_{\mathcal{L}})$. $\square$

Main Thm is proved.

Appendix

Resultant Let R be a ring. For $k \in \mathbb{N}$ let

$R_k[X] = \{P \in R[X], deg(P) \leq k\}$.

Then for $P, Q \in R[X]$ with degrees p and q consider the map :

$$\begin{array}{rccc} \Phi_{P,Q} : & R_{q-1}[X] \times R_{p-1}[X] & \longrightarrow & R_{p+q-1}[X] \\ & (S, T) & \mapsto & PS + QT \end{array}$$

Then $Res(P, Q) := det(\phi_{P,Q})$.

Claim. If $Res(P, Q) = 0$ then P and Q have a common root.

Hilbert Basis Theorem. If R is a noetherian ring then for

$n \geq 1$, $R[X_1, \ldots, X_n]$ is also noetherian.

Weierstrass Preparation Theorem. $f \in \mathbb{C}\{X_1, \ldots, X_n\}$ with

$f(0, \ldots, 0, X_n) = \alpha X_n^d + \textit{ higher terms}$, $\alpha \neq 0$.

$\Rightarrow \exists!\ u \in \mathbb{C}\{X_1, \ldots, X_n\},\ a_i \in \mathbb{C}\{X_1, \ldots, X_{n-1}\}$ s.t. :

$u(0) \neq 0,\ a_i(0) = 0$ and $f = u(X_{n+1}^d + a_1 X_{n+1}^{d-1} + \cdots + a_d)$.

Moreover $g \in \mathbb{C}\{X_1, \ldots, X_n\} \ \Rightarrow\ \exists!\ h \in \mathbb{C}\{X_1, \ldots, X_n\},\ b_i \in$

$\mathbb{C}\{X_1, \ldots, X_{n-1}\}$ s.t. : $g = hf + (b_1 X_n^{d-1} + \cdots + b_d)$.

Riemann Extension Theorem. $X \subset U \subset \mathbb{C}^n$ an analytic set

and f analytic on $U \setminus X$, bounded in a nbhd of x, $\forall x \in X$

$\Rightarrow f$ extends to an analytic function on U.

Mumford's Lemma. $f : X \to Y$ continuous between locally

compact top. spaces. $y \in Y$ s.t. $f^{-1}(Y)$ is compact.

$\Rightarrow \exists U \textit{ and } V$ open with $f(U) \subset V$ and $f : U \to V$ proper.