Chow's Theorem For Ideals

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Note. In the following U, V, W always denote open sets.

Definitions

Let X be a topological space. A **presheaf** $\mathcal{S}$ on X satisfies:

(i) For U open in X, $\mathcal{S}(U)$ is a commutative ring.

(ii) To each inclusion $V \subset U$, there correponds a morphism

$$res_{V,U} : \mathcal{S}(U) \to \mathcal{S}(V).$$

(iii) $res_{U,U} = id_{\mathcal{S}(U)}$.

(iv) $W \subset V \subset U \Rightarrow res_{W,V} \circ res_{V,U} = res_{W,U}$.

Moreover $\mathcal{S}$ is a **sheaf** and $(X, \mathcal{S})$ a **ringed space** if: Let $(U_i)_{i \in I}$ be an open covering of U and let $U_{i,j} := U_i \cap U_j$ for $i, j \in I$, then

(i) (Local identity) For $s, t \in \mathcal{S}(U)$, **sections** of $\mathcal{S}$ over U, if $s|_{U_i} = t|_{U_i}\ \forall i \in I$, then $s = t$.

(ii) (Gluing) If $(s_i) \in \prod U_i$ is such that $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$ for all $i, j \in I$, then $\exists s \in \mathcal{S}(U)$ s. th. $s|_{U_i} = s_i$.

Given a ringed space $(X, \mathcal{O}_X)$ a **sheaf of modules**

($\mathcal{O}_X$-modules), $\mathcal{S}$, is a sheaf s.th. $\mathcal{S}(U)$ is a module of $\mathcal{O}_X(U)$

and for $V \subset U$, $res_{V,U}$ is a morphism of modules.

For $x \in X$ we define the **stalk** at x, $\mathcal{S}_x$, as:

$$\mathcal{S}_x := \{(s, U), x \in U, s \in \mathcal{S}(U)\}/\sim$$

where $(s, U) \sim (t, V)$ if $\exists W \subset U \cap V$ s. th. $x \in W$ and

$s|_W = t|_W$. $\mathcal{S}_x$ comes with a natural module structure.

$\mathcal{S}$ is of **finite type** if for any $x \in X$, $\exists U \ni x$ s.th. $\exists s_1, \ldots, s_r$ sections of $\mathcal{S}$ over U s.th. for all $y \in U$, $\mathcal{S}_y$ is generated by $s_{1y}, \ldots, s_{ry}$.

$\mathcal{S}$ is of **relation finite type** if for any $U \subset X$, $n \in \mathbb{N}$ and any morphism $\psi : \mathcal{O}_X^n|_U \to \mathcal{S}|_U$, *ker* ψ is of finite type.

$\mathcal{S}$ is **coherent** if it is of finite type and relation finite type.

We will have particular interest in **sheaf of ideals**.

Setting for complex spaces.

For a complex space M, $\mathcal{O}_M$ is defined for an open set U as the ring of holomorphic functions on U. It is a coherent sheaf.

Let $f : M \to N$ be a holomorphic map between complex spaces, and $\mathcal{S}$ a sheaf on M. The **direct image** sheaf $f_*\mathcal{S}$ on N is given by $f_*\mathcal{S}(U) := \mathcal{S}(f^{-1}(U))$. It is a sheaf of $\mathcal{O}_N$-modules.

Fact 1. Direct Image Thm: f proper, $\mathcal{S}$ coherent $\Rightarrow f_*\mathcal{S}$ coherent.

Fact 2. If $\mathcal{S}$ is a sheaf on N, then there are a sheaf of ring $f'\mathcal{O}_N$ on M and a sheaf of $f'\mathcal{O}_N$-modules $f'\mathcal{S}$ with stalks $f'\mathcal{O}_{N,m} = \mathcal{O}_{N,f(m)}$ and $f'\mathcal{S}_m = \mathcal{S}_{f(m)}$, for $m \in M$.

Then the **pullback** $f^*\mathcal{S} := f'\mathcal{S} \otimes_{f'\mathcal{O}_N} \mathcal{O}_M$ is a sheaf of $\mathcal{O}_M$-modules. Note that $f^*\mathcal{O}_N = f'\mathcal{O}_N \otimes_{f'\mathcal{O}_N} \mathcal{O}_M \cong \mathcal{O}_M$.

If $\mathcal{I}$ is a sheaf of ideals on N then there is an induced injection $f^*\mathcal{I} \to f^*\mathcal{O}_N = \mathcal{O}_M$ and the image, $f^{-1}\mathcal{I}$, is a sheaf of ideals.

Fact 3. If $\mathcal{I}$ is coherent, then so is $f^{-1}\mathcal{I}$.

Blow-up of $\mathbb{C}^{n+1}$ at the origin.

Let $(y_0, \ldots, y_n)$ be coordinates on $\mathbb{C}^{n+1}$ and $[\xi_0 : \cdots : \xi_n]$ homogenous coordinates on $\mathbb{P}^n$ then the blow-up of $\mathbb{C}^{n+1}$ at 0 is the subset $\tilde{\mathbb{C}}^{n+1} \subset \mathbb{C}^{n+1} \times \mathbb{P}^n$ satisfying equations $y_i \xi_j = y_j \xi_i$.

$\pi_1 : \tilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ and $\pi_2 : \tilde{\mathbb{C}}^{n+1} \to \mathbb{P}^n$ are the holomorphic maps induced by projections.

Fact 4. If $\mathcal{I}$ is a sheaf of ideals on $\tilde{\mathbb{C}}^{n+1}$ then $\pi_{1*}\mathcal{I}$ is a sheaf of ideals on $\mathbb{C}^{n+1}$.

Chow's theorem for ideals.

Main Thm. Let U be an open nbhd of 0 in $\mathbb{C}^r$, X an analytic subset of $U \times \mathbb{P}^n$, $\mathcal{I}$ a coherent sheaf of $\mathcal{O}_X$-ideals on X. Then $\mathcal{I}$ is **relatively algebraic**, i.e. $\mathcal{I}$ is generated (after shrinking U if necessary) by a finite number of homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, with analytic coefficients in U-coordinates.

Rem 1. $\mathcal{I}$ may be considered as a coherent sheaf on $U \times \mathbb{P}^n$.

Fact 5. Oka coherence thm: The sheaf

$\mathcal{I}_X(V) = \{f \in \mathcal{O}(V), f|_{X\cap V} \equiv 0\}$ on an anlytic set X is coherent.

Rem 2. With Oka coherence thm the main thm (for U of dim=0)

implies Chow's thm.

Notations. Let $\sigma_1 := id_U \times \pi_1 : U \times \tilde{\mathbb{C}}^{n+1} \to U \times \mathbb{C}^{n+1}$ and

$\sigma_2 := id_U \times \pi_2 : U \times \tilde{\mathbb{C}}^{n+1} \to U \times \mathbb{P}^n$.

Then $\mathcal{J} := \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ is a coherent ideal sheaf on $U \times \mathbb{C}^{n+1}$.

Let $\tilde{\mathcal{I}} := \sigma_2^{-1}\mathcal{I}$ and $\tilde{\mathcal{J}} := \sigma_1^{-1}\mathcal{J}$, and note that $\tilde{\mathcal{J}} \subset \tilde{\mathcal{I}}$ in general.

Since σ_1 is biholomorhic off $\sigma_1^{-1}(U \times \{0\})$, $\tilde{\mathcal{J}} = \tilde{\mathcal{I}}$ in this domain.

Let $x = (x_1, \ldots, x_r)$ and $y = (y_0, \ldots, y_n)$ be coordinates on U and $\mathbb{C}^{n+1}$. If $F(x, y)$ is a holomorphic function in a nbhd of $(0, 0)$ in $U \times \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}^*$, let $F^{(\lambda)}(x, y) := F(x, \lambda y)$.

Lemma 1. $F \in \mathcal{J}_{(0,0)} \Longrightarrow F^{(\lambda)} \in \mathcal{J}_{(0,0)}, \forall \lambda \in \mathbb{C}^*$.

Proof. Let H be holomorphic in a nbhd of $(0,0)$, then $H \in \mathcal{J}_{(0,0)}$

iff $\sigma_1^* H$ is a section of $\tilde{\mathcal{I}}$ over some nbhd of $\sigma_1^{-1}(0,0) \cong \{0\} \times \mathbb{P}^n$

iff $\sigma_1^* H \in (\sigma_2^{-1}\mathcal{I})_p, \forall p \in \sigma_1^{-1}(0,0)$ (using properties of sheaves).

Let $p \in \sigma_1^{-1}(0,0)$, $q := \sigma_2(p)$ and $[\xi_0 : \cdots : \xi_n]$ homogeneous

coordinates on $\mathbb{P}^n$ s.th. $q = (0, [1 : 0 : \cdots : 0])$.

Let $W := \{\xi_0 \neq 0\}$, then $w_i := \xi_i/\xi_0$ are nonhomogeneous

coordinates on W. $\sigma_2^{-1}(U \times W) \cong U \times \mathbb{C} \times W$ is a nbhd of p in

$U \times \tilde{\mathbb{C}}^{n+1}$ with coordinates (x, y_0, w).

We have: $\sigma_1(x, y_0, w) = (x, y_0, y_0 w)$ and $\sigma_2(x, y_0, w) = (x, w)$.

$\mathcal{I}$ is coherent $\Rightarrow \exists\ G_1, \ldots, G_s$ generating $\mathcal{I}$ over a nbhd of q

$\Rightarrow \sigma_2^* G_1, \ldots, \sigma_2^* G_s$ generate $\tilde{\mathcal{I}}$ over a nbhd of $p = (0, 0, 0)$.

Since $\sigma_1^* F \in \tilde{\mathcal{J}} \subset \tilde{\mathcal{I}}$, $\exists\ A_1, \ldots, A_s$, holomorphic on a nbhd of p

s. th. $\sigma_1^* F(x, y_0, w) = \sum_i A_i(x, y_0, w) \sigma_2^* G_i(x, y_0, w)$.

Fix $\lambda \in \mathbb{C}^*$, then for y_0 small enough

$$\begin{aligned}
\sigma_1^* F^{(\lambda)}(x, y_0, w) &= \sum_i A_i(x, \lambda y_0, w) \sigma_2^* G_i(x, \lambda y_0, w) \\
&= \sum_i A_i(x, \lambda y_0, w) \sigma_2^* G_i(x, y_0, w)
\end{aligned}$$

So $\sigma_1^* F^{(\lambda)} \in (\tilde{\mathcal{I}})_p, \forall p \in \sigma_1^{-1}(0,0)$ and $F^{(\lambda)} \in \mathcal{J}_{(0,0)}$. $\square$

For F holomorphic on a nbhd of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ write:

$F(x,y) =: \sum_k \sum_{|\alpha|=k} a_\alpha(x) y^\alpha =: \sum_k F_k(x,y)$.

Lemma 2. $F^{(\lambda)} \in \mathcal{J}_{(0,0)}, \forall \lambda \in \mathbb{C}^* \Rightarrow F_k \in \mathcal{J}_{(0,0)}, \forall k \in \mathbb{N}$.

Proof. Let $A := \mathcal{O}_{U \times \mathbb{C}^{n+1},(0,0)}$. It is a Noetherian local ring. Let

$(y) := (y_0, \ldots, y_n)$ and $J := \mathcal{J}_{(0,0)}$ two ideals of A. For $\lambda \in \mathbb{C}^*$ let

$$Jet_m(F^{(\lambda)}) := \sum_{k=0}^{m} \lambda^k F_k$$

and note that $F^{(\lambda)} - Jet_m(F^{(\lambda)}) \in (y)^{m+1}$.

Note. $J = \cap_{m \geq m_0}(J + (y)^m), \forall m_0 \geq 0$ by a corollary of **Krull's Theorem**.

Since $Jet_m(F^{(\lambda)}) \in J + (y)^{m+1}$ for all $\lambda \in \mathbb{C}^*$, by taking $m+1$ different values for λ we get $F_k \in J + (y)^{m+1}$ for $k \leq m$.

Fix $k \in \mathbb{N}$, then $F_k \in \cap_{m \geq k+1}(J + (y)^m) = J$. $\square$

Consequently $\mathcal{J}_{(0,0)}$ is generated by elements of A homogeneous in y. Since A is Noetherian, $\mathcal{J}_{(0,0)}$ is generated by a finite number of these polynomials.

Due to the coherence of $\mathcal{J}$ they generate $\mathcal{J}$ over a nbhd of $(0,0)$. So we're left to prove:

Lemma 3. If $F_1, \ldots, F_s$ are homogeneous in y and generate $\mathcal{J}$ over a nbhd of $(0,0)$, they generate $\mathcal{I}$ over a nbhd of $\{0\} \times \mathbb{P}^n$.

Proof. It is enough to verify that $F_1, \ldots, F_s$ generate $\mathcal{I}$ on a nbhd of $q, \forall q \in \{0\} \times \mathbb{P}^n$ (by properties of sheaves).

Let $q \in \{0\} \times \mathbb{P}^n$ and ξ homogneneous coordinates on $\mathbb{P}^n$ s.th.

$q = (0, [1 : 0, \ldots, 0])$. Let w be the associated

nonhomogeneous coordinates on $W := \{\xi_0 \neq 0\}$.

We have local coordinates (x, y_0, w) on $\sigma_2^{-1}(U \times W)$ and:

$\sigma_1(x, y_0, w) = (x, y_0, y_0 w)$, $\sigma_2(x, y_0, w) = (x, w)$.

Let $G \in \mathcal{I}_q$, then $\sigma_2^* G$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a nbhd of

$\sigma_2^{-1}(q) = \{(0, y_0, 0), y_0 \in \mathbb{C}\}$.

$F_1, \ldots, F_s$ generate $\mathcal{J}$ on a nbhd of $(0,0) \Rightarrow \sigma_1^* F_1, \ldots, \sigma_1^* F_s$

generate $\tilde{\mathcal{J}}$ on a nbhd $V \subset U \times \tilde{\mathbb{C}}^{n+1}$ of $\sigma_1^{-1}(0,0)$.

This means that they generate $\tilde{\mathcal{I}}$ on $V \setminus \sigma_1^{-1}(U \times \{0\})$.

Pick $\tilde{y}_0 \neq 0$ s.th. $p := (0, \tilde{y}_0, 0) \in V$. Then $\sigma_2^* G \in \tilde{\mathcal{I}}_p = \tilde{\mathcal{J}}_p$, so $\exists$

$A_1, \ldots, A_s$ holomorphic on a nbhd of p s. th.

$$\sigma_2^* G(x, y_0, w) = \sum_{i=1}^{s} A_i(x, y_0, w) \sigma_1^* F_i(x, y_0, w)$$

But $\sigma_2^* G(x, y_0, w) = G(x, w)$ and

$\sigma_1^* F_i(x, y_0, w) = F_i(x, y_0, y_0 w) = y_0^{d_i} F_i(x, 1, w)$ since F_i are

homogeneous.

So let $a_i(x, w) := \tilde{y}_0^{d_i} A_i(x, \tilde{y}_0, w)$. Then

$$G(x, w) = \sum_{i=1}^{s} a_i(x, w) F_i(x, 1, w)$$

and $F_1(x, \xi), \ldots, F_s(x, \xi)$ generate $\mathcal{I}$ on a nbhd of q. $\square$