Plucker's Formula

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Basic Notions

For $f \in \mathbb{C}[x, y]$ we define a curve $\mathbb{C} \supset C_0 := \{(x, y)|f(x, y) = 0\}$

$f(x, y) = f_k(x, y) + ... + f_0(x, y)$ with $f_i(x, y)$ homogenous.

$F(x, y, z) := f_k(x, y)z^{k-k} + ... + f_0(x, y)z^{k-0}$

is homogenous degree k .

$\mathbb{CP}^2 \supset C := \{[x : y : z]|F(x, y, z) = 0\}$ with $F(x, y, 1) = f(x, y)$.

Plucker's Formula

A connected $\mathbb{C}$ curve C is an $\mathbb{R}$ surface.

The number of handles is the genus.

$\{f(x, y) = 0\} = C_0$ $(x, y) \in \mathbb{C}^2$, $deg(C) := deg(f)$.

Given smooth curve with genus g and degree d

Theorem Plucker's Formula: $g = \frac{(d-1)(d-2)}{2}$.

$f : C \longrightarrow C'$ holomorphic

Suppose $p \in C$ then $f(z) = z^m h(z)$ for some $h(z)$ in local coor.

Define: $v_p(f) := m$, $deg(f) := \sum_{p \in f^{-1}(q)} v_p(f) \; \forall q \in C'$.

Note: $v_p(f) = v_p(f') + 1$. $v_p(f) > 1$.

f(p) is branch, p ramification.

For compact C there are finite ramification points.

Hurwitz Formula

Thm: $2g - 2 = deg(f)(2g' - 2) + \sum_{p \in C}(v_p(f) - 1)$.

Proof: Triangulate C' s. th. branch points are vertices.

v' vertices, e' edges, t' faces.

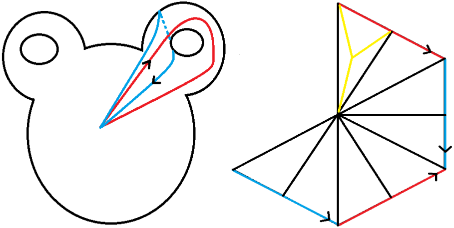
Lift triangulation via f to C . $e = deg(f)e'$, $t = deg(f)t'$.

$|f^{-1}(q)| = deg(f) + \sum_{p \in f^{-1}(q)}(1 - v_p(f))$

$\Rightarrow v = deg(f) \cdot v' - \sum_{p \in C}(v_p(f) - 1)$.

Hurwitz formula continued

Fact: $2 - 2g = v + t - e$.



v=2g+2 e=4g+8g t=8g

$$\Rightarrow 2 - 2g = v + t - e = deg(f)(v' + t' - e') - \sum_{p \in C}(v_p(f) - 1)$$

$$= deg(f)(2 - 2g') - \sum_{p \in C}(v_p(f) - 1) \quad \square$$

Description of Normalization

Local Normalization: Let f irreducible Weirstrass polynomial i.e.

$f(x,y) = y^k + a_1(x)y^{k-1} + \cdots + a_k(x) = \prod_i^k (y - y_i(x))$, $y_i \in \mathcal{O}$.

let $C = \{f(x,y) = 0\}$ Then: $\exists\ V \subset \mathbb{C}\ U \subset C$ both nbh. of 0

and $\exists\ g : V \to \mathbb{C}^2\ g : t \to (t^k, y_i(t^k))$ s. th. g is holomorphic

and g is biholomorphic on $V \backslash \{0\}$ onto $U \backslash \{0\}$.

Global Normalization

Given an algebraic Curve C , $\exists \tilde{C}$ with no singular points

and $\exists\ \sigma : \tilde{C} \to C$ holomorphic such that

1)$\sigma(\tilde{C}) = C$, 2) $\sigma^{-1}(sing(C))$ finite,

3) $\sigma|_{\tilde{C}\setminus\sigma^{-1}(sing(C))}$ is biholomorphic.

Can be done by essentially putting together local

normalizations for each singular point, and component.

Plucker's formula for C with no Singular pt.

$C_0 := \{f(x,y)=0\}$ f irreducible, $E_0 := \{g(x,y)=0\}$

$E_0, C_0 \subset \mathbb{C}^2$, $(0,0) \in E_0 \bigcap C_0$

if h is a local normalization of f, i.e. $h : t \to (t^k, v(t^k))$

$(C \cdot E)_0 := v_0(g \circ h)$, (Intersection number at 0)

$(C \cdot E)_p := (C \cdot E)_0$ translating p to 0

$(C \cdot E) := \sum_{p \in C \bigcap E} (C \cdot E)_p$ (Intersection number).

Proof of Plucker's Formula

Let $C_0 := \{f(x,y)=0\}$, $E_0 := \{\frac{\partial}{\partial y} f(x,y)=0\}$

Let $F(x,y,z)$ homog. of f . $deg(F)=d \Rightarrow deg(\frac{\partial}{\partial y}F)=d-1$

Calculate $(C \cdot E)$ in two ways;

First by Bezoit's Theorem $(C \cdot E)=d(d-1)$.

Second we can find coordinate in $\mathbb{CP}^2$ s. th. $|C \bigcap L_\infty| = d$.

Set $\pi : C \to \mathbb{CP}^1$ via $[x:y:z] \to [x:z]$, $deg(\pi)=d$.

Lemma 1: $p \in C \bigcap E \Rightarrow p$ is a ram. pt. of π

Also $(C \cdot E)_p = v_p(\pi) - 1$.

Proof: $\frac{\partial}{\partial y} f = 0 \Rightarrow \frac{\partial}{\partial x} f \neq 0$ (no sing. pts. on C)

By IFT $f(x(y), y) = 0$ in a nbh. of p

$\Rightarrow \frac{\partial}{\partial x} f(x(y), y) x'(y) + \frac{\partial}{\partial y} f(x(y), y) = 0$

$\Rightarrow (C \cdot E)_p = ord(\frac{\partial}{\partial y} f(x(y), y))$

$= ord(x'(y)) = ord(x(y)) - 1 = v_p(\pi) - 1 \quad \square$

Lemma 2: p is a ram. pt. of $\pi \Rightarrow p \in C \bigcap E$

Proof: Suppose $(\frac{\partial}{\partial y} f(p) \neq 0)$ then by IVF theorem $y = y(x)$ $\square$

Combining the lemmas we have:

$(C \cdot E)$=$\sum_{p \in ram(\pi)} (C \cdot E)_p = \sum_{p \in ram(\pi)} (v_p(\pi) - 1) =$

$\sum_{p \in C} (v_p(\pi) - 1)$. By Hurwirtz $(C \cdot E) = 2(g + d - 1)$

Solving with Bezout $g = \frac{(d-1)(d-2)}{2}$ $\square$

Plucker with double points

Suppose C has n ordinary double points.

Let $\tilde{C}$ be normalization of C , $g :=$ genus of $\tilde{C}$

$g = \frac{(d-1)(d-2)}{2} - n$

Proof: WLOG $|\tilde{C} \bigcap L_\infty| = d$

and sing. pt. have non-vertical tangents.

Define new map $\gamma = \pi \circ \sigma$, $deg \gamma = d$.

$(C \cdot E)_p$ for p singular

As before $p \in (E \bigcap C) \backslash Sing(C) \Rightarrow (C \cdot E)_p = v_p(\gamma) - 1$

$p \in Sing(C)$ translate to $(0, 0)$

$\Rightarrow f(x, y) = ax^2 + 2bxy + cy^2 + f_3(x, y) + ...$

and $\frac{\partial}{\partial y} f(x, y) = 2bx + 2cy$. No vert. tang. in $Sing(C)$

$\Rightarrow ac - b^2 \neq 0$, $c \neq 0$. Using IFT on $\frac{\partial}{\partial y} f(x, y) = 0$

$y = y(x) = -\frac{b}{c}x + ...$ (Taylor Expansion)

Conclusion

$$f(x, y(x)) = \frac{ac-b^2}{c}x^2 + ... \Rightarrow (C \cdot E)_p = 2\,.$$

$$(C \cdot E) = \sum_{p \in E \bigcap C} (C \cdot E)_p$$

$$= \sum_{p \in ram(\pi) \setminus Sing(C)} (v_p(\pi) - 1) + \sum_{p \in Sing(C)} (C \cdot E)_p$$

$$= \sum_{p \in C} (v_p(\pi) - 1) + 2n\,.$$

Plugging in Hurwitz and equating to Bezout

$$g = \frac{(d-1)(d-2)}{2} - n \quad \square$$