# Weierstrass Preparation Theorem, Resultants and Puiseux Factorization Theorem

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# Special Weierstrass Div. Thm(S.W.D.) $\Rightarrow$ W.D.Thm

$\forall\ f \in \mathbb{C}\{x,y\}$ (conv. power series) and $P^d(\lambda, y) := y^d + \sum_{k=1}^{d} \lambda_k y^{d-k}$

exists $Q_f^d(x,\lambda,y) \in \mathbb{C}\{x,\lambda,y\}$ and $R_f^d(x,\lambda) \in \mathbb{C}\{x,\lambda\}$ s. th.

$$f(x,y) = P^d(\lambda,y) Q_f^d(x,\lambda,y) + \sum_{j=1}^{d} R_{f,j}^d(x,\lambda) y^{d-j} \ . \qquad (\diamond)$$

**Weier. Prep. Thm (W.P.Thm.)**: If $f(0,y) = y^d h(y)$, $h(0) \neq 0 \Longrightarrow$

exist unique $\{\lambda_k(x)\}_k$ , $\lambda_k(0) = 0$ and $Q_f \in \mathbb{C}\{x,y\}$ s.th. $Q_f(0,0) \neq 0$

and $$f(x,y) = P^d(\lambda(x),y) Q_f(x,y) \ .$$

**W.D.Thm**: Division of $g(x,y) \in \mathbb{C}\{x,y\}$ by f with remainder of $deg_y < d$

## Special W.D.Thm $\Rightarrow$ W.P.Thm. Pick $Q_f^d(x,\lambda,y)$ and $R_{f,j}^d(x,\lambda)$

**Proof**: Plug in $x=0$, $\lambda=0$ in eqn. $(\diamond) \Longrightarrow$

$$y^d h(y) := y^d Q_f^d(0,0,y) + \sum_{j=1}^{d} R_{f,j}^d(0,0) y^{d-j}$$

$\Longrightarrow$ all $R_{f,j}^d(0,0)=0$ , $Q_f^d(0,0,y)=h(y)$ and $Q_f^d(0,0,0)\neq 0$ .

Plug in $x=0$ , $\lambda=0$ in $\frac{\partial}{\partial\lambda_i}$ eqn. $(\diamond) \Longrightarrow$

$$0 \equiv y^{d-i} Q_f^d(0,0,y) + y^d \frac{\partial Q_f^d}{\partial\lambda_i}(0,0,y) + \sum_{j=1}^{d} \frac{\partial R_{f,j}^d}{\partial\lambda_i}(0,0) y^{d-j}$$

$\Rightarrow \forall\ j>i\ \frac{\partial R_{f,j}^d}{\partial\lambda_i}(0,0)=0$ , $\frac{\partial R_{f,i}^d}{\partial\lambda_i}(0,0)=-h(0) \Rightarrow det(\frac{\partial R_{f,j}^d}{\partial\lambda_i})(0,0)\neq 0$ .

$\overset{I.F.Thm.}{\Longrightarrow} \exists\ \lambda_j(x)\in\mathbb{C}\{x\}$ , s.th. $\lambda(0)=0$ , $R_{f,j}^d(x,\lambda(x))\equiv 0$ .

# Almost W.D.Thm.(Newton Inter.):Say $f \in \mathbb{C}\{x, y\}$ and

$p^d(\mu^d, y) := \prod_{1 \le i \le d} (y - \mu_i)$ , where $\mu^d := (\mu_1, \ldots, \mu_d)$ . Then

$$f(y) := f(\cdot, y) = p^d(\mu^d, y) q^d(x, \mu^d, y) + \sum_{j=1}^{d} r_j^d(x, \mu^d) y^{d-j} \qquad (\star)^d$$

with $q^d \in \mathbb{C}\{x, \mu^d, y\}$ and $r_j^d \in \mathbb{C}\{x, \mu^d\}$ . (We skip writing x.)

**Proof** Obvious for $d = 1$ : $f(y) = (y - \mu_1)\frac{f(y) - f(\mu_1)}{(y - \mu_1)} + f(\mu_1)$ .

Let $\Delta_y^1 f(x; \mu_1, y) := \frac{f(y) - f(\mu_1)}{(y - \mu_1)}$ , $(\Delta_y^{d+1} f) := \Delta_y^1(\Delta_y^d)(\mu^d; \mu_{d+1}, y)$ .

Note $\Delta_y^1 f(x; \mu_1, y) = \int_0^1 \frac{\partial f}{\partial y}(x, \tau y + (1 - \tau)\mu_1) d\tau \in \mathbb{C}\{x, \mu_1, y\}$ .

# Proof of Newton Interpolation: Suffices $(\star)^d \Rightarrow (\star)^{d+1}$

Apply $d = 1$ case to $q^d(\mu^d, y)$ with respect to y and plug into $(\star)^d$

$$q^d(\mu^d, y) = (y - \mu_{d+1})\Delta_y^1 q^d(\mu^d, \mu_{d+1}, y) + q^d(\mu^d, \mu_{d+1})$$

$\Longrightarrow (\star)^{d+1}$ with $q^{d+1}(\mu^{d+1}, y) = (\Delta_y^{d+1} f)(\mu^{d+1}, y)$ and

$$\sum_{j=1}^{d+1} r_j^{d+1}(\mu^{d+1}) y^{d+1-j} := \sum_{j=1}^{d} r_j^d(\mu^{d+1}) y^{d-j} + \prod_{i=1}^{d} (y - \mu_i)(\Delta_y^d f)(\mu^d, \mu_{d+1}) \, .$$

# All Newton's divided differences $\Delta_y^d f(\mu^{d+1})$ are symmetric!

**Proof:** Induction on $d$ and the case of $d = 2$ suffice:

Clearly $$\Delta_y^2 f(\mu^3) = \frac{\dfrac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - \dfrac{f(\mu_3) - f(\mu_2)}{\mu_3 - \mu_2}}{\mu_1 - \mu_3}$$

is symmetric in $\mu_1, \mu_2, \mu_3$. For induction replace $f$ by $\Delta_y^{d-2} f$ .

**Cor** All $r_j^d(\mu^d)$ of the Almost W.D.Th. are symmetric with respect to $\mu^d$ :

Since $p^d, f$ and $q^d$ are symmetric in $\mu^d$ , so is $r_j^d(\mu^d)$ .

# $P^d(\sigma(\mu^d), y) := p^d(\mu^d, y)$ defines Elem. symm. poly.'s

**Thm 1** $\mathcal{SP}_k := \{\text{hom. symm. pol. of deg. } k\} = \sigma^*(\mathcal{W}H_k)$ , where

$$\mathcal{W}H_k := \{W_k : W_k(\sigma) = \textstyle\sum_{\beta:\sum_{j=1}^d j\beta_j = k} c_\beta \sigma_1^{\beta_1} \dots \sigma_d^{\beta_d}\}$$

(called weighted hom. pol. of deg. $k$ with weights wt $\sigma_i = i$ ).

**Cor 1**: Every symm. formal $\sum_{k=0}^\infty H_k(\mu) = \sum_{k=0}^\infty W_k(\sigma(\mu))$, $W_k \in \mathcal{W}H_k$

**Cor 2**: Weierstrass Div. Thm. version with formal expansion

$\hat{Q}^d(x, \sigma, y)|_{\sigma=\sigma(\mu)} := q^d(x, \mu, y)$ and $\hat{R}_j^d(x, \sigma)|_{\sigma=\sigma(\mu)} := r_j^d(x, \mu)$ .

# Claim 1: $ker\ \sigma^*|_{\mathcal{W}H_k} = \{0\}$

Denote $\sigma(\mu^d) := (\sigma_1(\mu^d), \ldots, \sigma_d(\mu^d))$ and $\sigma^* H(\mu^d) := H(\sigma(\mu^d))$ .

**Proof:** $H \in ker\ \sigma^*|_{\mathcal{W}H_k} \implies DH(\sigma(\mu)) \cdot \frac{\partial \sigma}{\partial \mu} \equiv 0.\ P^d(\sigma(\mu^d), \mu_j) \equiv 0$ .

Fix j and take partials $\frac{\partial}{\partial \mu_i}(P^d(\sigma(\mu^d), \mu_j) \equiv 0 \iff$ in matrix forum

$-T := -[\delta_{ij} \cdot \frac{\partial P^d}{\partial y}(\sigma(\mu^d), \mu_j)] \equiv [\mu_j^{d-k}] \cdot [\frac{\partial \sigma_k}{\partial \mu_i}(\mu^d)]$ and since

$det T \equiv (-1)^{d\cdot(d+1)/2} \cdot \prod_{i<j}(\mu_j - \mu_i)^2 \Rightarrow det(\frac{\partial \sigma_k}{\partial \mu_i}) \not\equiv 0 \Rightarrow \sigma^*(DH) \equiv 0$

$\implies$ inductively $\sigma^*(D^\alpha H) = 0) \Rightarrow (D^\alpha H)(0) = 0$ for all $\alpha$, i.e.

ker $\sigma^*|_{\mathcal{W}H_k} = \{0\}$ . (In fact $|det(\frac{\partial \sigma_k}{\partial \mu_i}| \equiv \prod_{i<j} |\mu_j - \mu_i|$ !)

## Claim 2: dim $S\mathcal{P}_k$ = dim $\mathcal{W}H_k$ (Note: $\sigma_i \in S\mathcal{P}_i$)

**Fact 1**: $\{\sigma^\alpha\}_{\sum_{1\le j\le d} j\alpha_j = k}$ form a basis of $\mathcal{W}H_k$ $(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d)$.

**Fact 2**: Consider $\mu^\beta$, $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d$ with $\beta_{j+1} \le \beta_j$ , $|\beta| = k$ .

Then symmetrizations $S(\mu^\beta)$ form a basis of $S\mathcal{P}_k$ .

Let $\Psi(\alpha) = \beta$, where $\beta_j = \alpha_j + \cdots + \alpha_d$ , $j \ge 1$ .

Then $\Psi$ is one-to-one and $|\Psi(\alpha)| = \alpha_1 + 2\alpha_2 + \cdots + d\alpha_d$ .

# W.D.Thm. (Formal) $\Longrightarrow$ W.D.Thm. (Convergent)

**Proof**: Via Thm: If $\sigma^* G(\mu) \in \mathbb{C}\{\mu\} \Rightarrow G(\sigma) \in \mathbb{C}\{\mu\}$ (I owe you).

$\Rightarrow \sigma^* Q_f^d(\sigma) := q^d(\mu)$ , $\sigma^*(R_{f,j}^d) := r_j^d \Rightarrow Q_f^d \in \mathbb{C}\{x, \sigma, y\}$ , $R_{f,j}^d \mathbb{C}\{x, \sigma\}$ .

$$\sigma^*\{f(x,y) - [P^d(\sigma,y)Q_f^d(\sigma,y) + \sum_{k=1}^{d} R_{f,k}^d(x,\sigma)y^{d-k}]\} = 0 \text{ . } ker\ \sigma^* = \{0\}$$

$$\Rightarrow f(x,y) = P^d(\sigma,y)Q_f^d(\sigma,y) + \sum_{k=1}^{d} R_{f,k}^d(x,\sigma)y^{d-k} \text{ W.D.Thm(conv).}$$

**Corollary**: $\mathbb{C}\{x\}$ is a UFD.

Via W.P.T. and Gauss Thm: $R$ is UFD $\Rightarrow R[y]$ is UFD. (I owe you.)

# Resultants: Say $P^d(c(\lambda,\mu),y) := P^q(\lambda,y)\cdot P^{d-q}(\mu,y)$

and let $A(\lambda,y) := P^q(\lambda,y)$ , $B(\mu,y) := P^{d-q}(\mu,y)$ . Note:

$\sum_k \frac{\partial c_k}{\partial\lambda_i}(\lambda,\mu)y^{d-k} = y^{q-i}B(\mu,y)$ , $\sum_k \frac{\partial c_k}{\partial\mu_j}(\lambda,\mu)y^{d-k} = y^{d-q-j}A(\lambda,y)$ .

Define linear map $Res_{A,B}(\lambda,\mu) : (Pol_{q-1}\oplus Pol_{d-q-1}) \longrightarrow Pol_{d-1}$ ,

by matrix $\frac{\partial c}{\partial(\lambda,\mu)}$ (in basis $y^j$) ; i.e. $(F,G)\mapsto F\cdot B(\mu,y)+G\cdot A(\lambda,y)$

**Thm.** $\nexists$ a com. rt. of $A(\lambda^0,y)$ and $B(\mu^0,y) \Leftrightarrow det\ Res_{A,B}(\lambda^0,\mu^0)\neq 0$ .

**Proof**: Since both $\Leftrightarrow$ $\nexists$ non-triv. sol. to $F\cdot B(\mu^0,y)+G\cdot A(\lambda^0,y)\equiv 0$.

**Cor. (RC)** $\exists\ \lambda^*,\mu^*\in\mathbb{C}\{c-c(\lambda^0,\mu^0)\}$ s.th. $p^d(c,y)=A(\lambda^*,y)B(\mu^*,y)$

## Newton-Puiseux Thm(NP): Say $f = p^d(\lambda(x), y)$ , $\lambda(0) = 0$ and

$V(f) := \{f(x,y) = 0\} \subseteq (\mathbb{C}^2, 0)$ . Then

1. $\exists k \in \mathbb{N}$ and $y(t) \in \mathbb{C}\{t\}$ s. th. $y(0) = 0$ and $f(t^k, y(t)) = 0$ ;

2. If $f(x,y)$ is irred. in $\mathbb{C}\{x\}[y]$ then

$$f(t^d, y) = \prod_{j=1}^{d}(y - y(\epsilon^j t)) ,$$

where $\epsilon = e^{2\pi i/d}$ is the primitive root of $\epsilon^d = 1$ .

**Cor.** Can describe $V(f) \subseteq (\mathbb{C}^2, 0)$ for $f \in \mathbb{C}\{x,y\}$ : WP then NP Thm.'s.

## Lemma: $f(x,y) = y^d + \sum_{i=1}^{d} c_i(x) y^{d-i}$, $c_i(x) \in \mathbb{C}\{x\}$ .

If for a $p \in \mathbb{Z}_+$ all $x^{ip}$ divide $c_i(x)$ then define $d_i(x) := c_i(x)/x^{ip}$ and

$$g(x,y) := y^d + \sum_{i=1}^{d} d_i(x) y^{d-i} \; .$$

Then f is irred. $\Rightarrow g(0,y) = (y+a)^d$ .

**Proof:** Otherwise $g(x,y)$ factors in $\mathbb{C}\{x\}[y]$ (by Cor. RC).

Let $\mu_0(h)$ be the largest int. $\mu$ s.th $x^{\mu}$ divides $h \in \mathbb{C}\{x\}$, $\infty$ if $h \equiv 0$ .

**Cor.** If $c_1(x) \equiv 0$ for $f$ as before and $p = min_{2 \le i \le d} \mu_0(c_i)/i \in \mathbb{N}$ then

## show $f$ is not irreducible. Followed by proof of NP 1 .

**Proof** If f is irred. by prev. $g(0, y) = (y + a)^d \Rightarrow a = 0$, but $p < \infty \Rightarrow$ ?!

**Proof of 1. in NP Thm.** We may assume that f is irreducible.

By translation $y' = y + c_1/d$ may assume $c_1 \equiv 0$ (then $c_i \not\equiv 0$ for some $i$) .

Also $min_{2 \leq i \leq d}(\mu_0(c_i)/i) =: p/q$, where $gcd(p, q) = 1$ . Then $q > 1$ .

Since $\mu_0(c_i(t^q)) = q \cdot \mu_0(c_i(t)) \Rightarrow f_1(t, y) = f(t^q, y)$ is reducible $\Rightarrow$

(induction) $\exists\ k', y(t) \in \mathbb{C}\{t\}$ s.th. $y(0) = 0$ and $f_1(t^{k'}, y(t)) = 0$,

so letting $k = qk'$ completes the proof.

# Proof of 2 in NP Thm.

Choose $k$ , $y(t) = \sum_{j=1}^{\infty} a_j t^{k_j}$ s.th. $f(t^k, y(t)) = 0$

and $k, k_1, \dots$ are positive integers with no common divisor $\Rightarrow$

$f(t^k, y) = \prod_{j=1}^{k} (y - y(\epsilon^j t)) g(t, y)$ (Since $y(\epsilon^j t)$ are distinct) .

**Claim:** If f is irreducible then $g(t, y) \equiv 1$ and $k = d$.

Suffices to show that $\prod_{j=1}^{k} (y - y(\epsilon^j t)) \in \mathbb{C}\{t^k\}[y]$. Each coefficient

of $t^r y^s$ is a linear comb. of terms $\sigma^\alpha := \prod_{wt(\alpha)=r} \sigma_i(\mu^k)^{\alpha_i}$ evaluated at

$\mu^k = (\epsilon, \epsilon^2, \dots, \epsilon^k)$, i.e. $P^k(\sigma, y) = y^k - 1 \Rightarrow \sigma^\alpha = 0$ unless $r \in k \cdot \mathbb{Z}$.