Puiseux Exponents, Exponent of contact and Equisingular curves.

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Review: W.D.T: $\forall f \in \mathbb{C}\{x,y\}$ and $P^d(\lambda, y) := y^d + \sum_{k=1}^{d} \lambda_k y^{d-k}$

exists $Q_f^d(x,\lambda,y) \in \mathbb{C}\{x,\lambda,y\}$ and $R_f^d(x,\lambda) \in \mathbb{C}\{x,\lambda\}$ s. th.

$$f(x,y) = P^d(\lambda,y) Q_f^d(x,\lambda,y) + \sum_{j=1}^{d} R_{f,j}^d(x,\lambda) y^{d-j} \ .$$

Weier. Prep. Thm (W.P.Thm.): If $f(0,y) = y^d h(y)$, $h(0) \neq 0 \Longrightarrow$

exist unique $\{\lambda_k(x)\}_k$, $\lambda_k(0) = 0$ and $Q_f \in \mathbb{C}\{x,y\}$ s.th. $Q_f(0,0) \neq 0$

and $$f(x,y) = P^d(\lambda(x),y) Q_f(x,y) \ .$$

Newton-Puiseux Thm(NP): Say $f = p^d(\lambda(x), y)$, $\lambda(0) = 0$ and

$V(f) := \{f(x, y) = 0\} \subseteq (\mathbb{C}^2, 0)$. Let $\eta_d := e^{2\pi i/d}$. Then

1. $\exists k \in \mathbb{N}$ and $h(t) \in \mathbb{C}\{t\}$ s. th. $h(0) = 0$ and $f(t^k, h(t)) = 0$;

2. If $f(x, y)$ is irred. in $\mathbb{C}\{x\}[y] \Rightarrow f(t^d, y) = \prod_{j=1}^{d}(y - h(\epsilon^j t))$,

where $\epsilon = \eta_d$ is the primitive root of $\epsilon^d = 1$, and $(t^d, h(t))$

is a **desingularization** of $V(f)$. Let $(h)_j(t) := h(\epsilon^j t)$.

Cor: Can describe $V(f) \subseteq (\mathbb{C}^2, 0)$ for $f \in \mathbb{C}\{x, y\}$: WP then NP Thm.'s.

Def. Let $f := \prod f_i$, where f_i are irred. then $V(f_i)$ are **branches** of $V(f)$.

Fact 1: $F(x) := \sum F_i x^i \in \mathbb{C}\{x\}^*$ iff $F_0 \in \mathbb{C}^*$ (Clear) .

Prop 1: $\mathbb{C}\{\vec{x}^n\}$, where $\vec{x}^n := (x_1, \ldots, x_n)$, is a UFD.

Proof: In 1 var., the only prime is x $\Rightarrow \mathbb{C}\{\vec{x}^1\}$ is a UFD. Induction on n.

If $F \in \mathbb{C}\{\vec{x}^n, y\}$, $F(0, y) = y^d \cdot h(y)$, $h(0) \neq 0 \overset{W.P.Thm}{\Longrightarrow} F = Q_F \cdot G$,

where Q_f is a unit and $G \in \mathbb{C}\{\vec{x}^n\}[y]$.

Notation: $\mathcal{A}_{n+1} := \mathbb{C}\{\vec{x}^n, y\}$, $\mathcal{PA}_n := \mathbb{C}\{\vec{x}^n\}[y]$. Say $G \in \mathcal{PA}_n$.

$\mathcal{PA}_n$ UFD (Gauss) $\Rightarrow \mathcal{A}_{n+1}$ UFD. Let $G = \prod G_i$

with G_i is irred. in $\mathcal{A}_{n+1}$: Otherwise let $G_i = H_1 \cdot H_2$ and $H_i(0, y) \not\equiv 0$

$\overset{W.P.Thm}{\Longrightarrow} H_1 = Q_1 \cdot H_1'$ and $H_2 = Q_2 \cdot H_2' \Rightarrow G_i = (Q_1 \cdot Q_2) \cdot H_1' \cdot H_2'$?! .

Uniqueness: Suffices to show ideal $(G_i) \hookrightarrow \mathcal{A}_{n+1}$ is prime.

If G_i divides $H_1 \cdot H_2 \overset{WDT}{\Rightarrow} H_1 = G_i \cdot Q_1 + R_1$ and $H_2 = G_i \cdot Q_2 + R_2$

$\Rightarrow G_i$ divides, in $\mathcal{A}_{n+1}$, $R := R_1 \cdot R_2 \in \mathcal{PA}_n \Rightarrow$ divides in $\mathcal{PA}_n$: since

$R = G_i \cdot q + r$ with unique $q, r \in \mathcal{PA}_n$, $deg\ r < deg\ G_i$, but $r = 0$ due

to W.D.Thm $\Rightarrow G_i$ divides R_1 or R_2 in $\mathcal{PA}_n$ (hence H_1 or H_2 in $\mathcal{A}_{n+1}$) $\square$

Intersection Number: Let $\mathbf{0} := (0,0) \in \Gamma_1 := V(g) \subset \mathbb{C}^2$,

$g =: \sum g_{ij} x^i y^j \in \mathcal{A}_2$, $\Gamma_2 := Im(\psi : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0}))$, $\psi \in hol.$ and

$(\Gamma_1.\Gamma_2)_{\mathbf{0}} := ord(g \circ \psi)$, $m := m_{\mathbf{0}}(\Gamma_1) := ord_{\mathbf{0}} g$ and $g_m := \sum_{i+j=m} g_{ij} x^i y^j$.

Lemma 1: L is a lin. factor of $g_m(x,y)$ iff $(L.\Gamma_1)_0 > m_{\mathbf{0}}(\Gamma_1)$.

Proof: If L does not divide $g_m \Rightarrow ord_x(g(x,ax)) = m_{\mathbf{0}}(\Gamma_1)$.

If L divides $g_m \Rightarrow g_m(x,ax) = 0 \Rightarrow ord_x(g(x,ax)) > m_{\mathbf{0}}(\Gamma_1)$. □

Def.: Tangent lines to $V(g)$ are the $V(L)$ for L that divide g_m .

Puiseux Exponents: Γ irred. at $\mathbf{0}$, choose coord s.th.

x-axis not tang. to $\Gamma \Rightarrow \exists$ desing. $y(t) := \sum_{k=1}^{\infty} a_k t^{n_k}$, where $a_k \neq 0$,

$n_k > n_{k-1}$ and $n_1 \geq n_0 := m_0(\Gamma)$. Set $d_i := gcd(n_0, n_1, \ldots, n_{k_i})$

with k_i s.th. $d_i < d_{i-1}$, $d_0 := n_0$.

Then **Puiseux Exponents** are $P(\Gamma) := \{n_{k_i}/n_0\}_{1 \leq i}$, $D(\Gamma) := \{d_i\}_{0 \leq i}$.

Let $B_i = V(g_i)$, g_i irred. , $m_i := ord_{\mathbf{0}} g_i$, $d := lcm(m_1, m_2)$, $\ell_i = d/m_i$,

and $\epsilon := \eta_d$, $\epsilon_i := \eta_{m_i}$, i= 1 , 2 .

$g_i(t^{m_i}, y) = \prod_{1 \leq j \leq m_i} (y - h_i(\epsilon_i^j t))$ NPT $\Rightarrow g_i(t^d, y) = \prod_{1 \leq j \leq m_i} (y - h_i((\epsilon^j t)^{\ell_i}))$

Exp. of contact: $\mathcal{O}(h_1, h_2) := \frac{1}{d} ord_t(h_1(t^{\ell_1}) - h_2(t^{\ell_2}))$.

$ord_x(h_1(x^{\frac{1}{m_1}}) - h_2(x^{\frac{1}{m_2}})) := \mathcal{O}(h_1, h_2)$. Let $\mathcal{LN}(B_i) := \{(h_i)_j(t)\}_{j\in[1,m_i]}$.

Lemma 2: Set $s := \{\mathcal{O}((h_1)_j, h_2) : (h_1)_j \in \mathcal{LN}(B_1)\}$ depends only on

B_1 , B_2 for any choice of $h_2(t)$. Denote $\mathcal{S}(B_1, B_2) := s$. Note $\epsilon^{\ell_i} = \epsilon_i$.

Proof: Recall $(h_2)_j(t^{\ell_2}) = h_2((\epsilon^j t)^{\ell_2})$, $t \to \epsilon^j t$ keeps $\mathcal{LN}(B_i)$ invar. □

Say $\mathcal{O}(B_1, B_2) := max\ \mathcal{S}(B_1, B_2)$, $\{\alpha_i\}_{i\geq 1} := P(B_1)$, $\kappa := \mathcal{O}(B_1, B_2)$.

Prop. 2: Then $\alpha_q \leq \kappa \leq \alpha_{q+1}$ for some q ; α_i for $i = 1, 2, \ldots, q$,

appears as $\mathcal{O}((h_1)_j, h_2)$ exactly $d_{i-1} - d_i$ and κ exactly d_q times.

Proof: Say $\mathcal{O}(h_1, h_2) = \kappa \Rightarrow \mathcal{LN}(B_1) := \{h_1(\epsilon_1^j t)\}_{1\leq j\leq m_1}$,

where $\epsilon_1^{m_1} = 1$, each $\alpha_i = ord_t(h_1(t) - h_1(\epsilon_1^{j_0} t))$ iff

$j_0 \in \{j : \frac{j\cdot\alpha_{i-1}}{m_1} \in \mathbb{Z}, \frac{j\cdot\alpha_i}{m_1} \notin \mathbb{Z}\} =: \mathcal{S}_i$ and $\#\mathcal{S}_i = d_{i-1} - d_i$.

Prop. 3: $(B_1.B_2)_0 = \sum_{h_1\in\mathcal{LN}(B_1), h_2\in\mathcal{LN}(B_2)} \mathcal{O}(h_1, h_2)$.

Proof: By NP Thm $g_1(t^d, y) =: \prod_{r=1}^{m_1}(y - (h_1)(\epsilon^r t)^{\ell_1})$.

Substituting we have $g_1(t^d, h_2(t^{\ell_2})) = \prod_{r=1}^{m_1}(h_2(t^{\ell_2}) - h_1(\epsilon^r t)^{\ell_1})$,

$\Rightarrow ord_t(g_1(t^d, h_2(t^{\ell_2}))) = \sum_1^{m_1} ord_t(h_2(t^{\ell_2}) - h_1((\epsilon^r t)^{\ell_1}))$,

$\Rightarrow (B_1.B_2)_0 = m_2 \sum_{h_1\in\mathcal{LN}(B_1)} \mathcal{O}(h_1, h_2) = \sum_{h_i\in\mathcal{LN}(B_i)} \mathcal{O}(h_1, h_2)$. □

Let $\Gamma_k = V(g_k)$, $g_k \in \mathbb{C}\{x, y\}$, $k = 0, 1$.

Def: $\Gamma_0 \underset{alg.}{\approx} \Gamma_1$, when $\exists$ a bijection between their branches $B_{0i} \leftrightarrow B_{1i}$ s.th.

$P(B_{ki})$ and $\mathcal{O}(B_{ki}, B_{kj})$ is independent of k .

Def: Let $Cyl_\rho := \{(x, y) \in \mathbb{C}^2 : |x| = \rho, |y| \leq \rho\}$ the **Link of** Γ at **0** is

$K(\Gamma) := \Gamma \cap Cyl_\rho \hookrightarrow Cyl_\rho$ for 'small' $\rho > 0$. If Γ is irred., $K(\Gamma)$ is a **knot**.

Fact 2: For two knots $K(B)$, $K(B')$ the linking number is $(B.B')_{\mathbf{0}}$.

Def: $\Gamma_0 \underset{top}{\approx} \Gamma_1$, when there is a C^0 map $H : Cyl_\rho \times [0, 1] \to Cyl_\rho$, s.th.

$H(x, t_0)$ are homeo., $H(x, 0)$ is the identity, and $H(K(\Gamma_0), 1) = K(\Gamma_1)$.

Main Theorem: $\Gamma_0 \underset{alg.}{\approx} \Gamma_1 \Leftrightarrow \Gamma_0 \underset{top}{\approx} \Gamma_1$ (we show $\Rightarrow$) .

Lem. 3 (Isotopy Extension Theorem): Let X and Y be compact smooth

manf.s , $F : X \times [0,1] \to Y$ a smooth map, $F(x,t) =: f_t(x)$, s.th.

each $f_t(x)$ is a smooth embedding. Then $\exists\ \hat{H} := Y \times [0,1] \to Y \times [0,1]$

a diffeomorphism, s.th. $f_t(x) = h_t(f_0(x))$, where $(h_t(y), t) := \hat{H}(y,t)$.

Proof: Let $\xi := \sum_j \frac{\partial F_j}{\partial t} \frac{\partial}{\partial y_j} + \frac{\partial}{\partial t}$, at $(y = f_t(x), t)$, $\forall\ (x,t) \in X \times [0,1]$,

otherwise require component in $\frac{\partial}{\partial t}$ direction be 1 $\Rightarrow \exists\ \hat{H}$, ODE's $\square$

Lem. 4: $P(B) = P(B') \Rightarrow$ K(B) and $K(B')$ isotopic knots.

Let $\zeta := (\rho e^{im_0(B)\theta}, \sum_{r=m_0(B)\theta}^{\infty} a_r \rho^{r/m_0(B)} e^{ir\theta})$ parameterize K(B) by θ ,

$\zeta_s := (\rho e^{im\theta}, \sum_{r=m}^{\infty} a_r(s) \rho^{r/m} e^{ir\theta})$, $s \in [0,1]$, where $m := m_0(B)$,

and $a_r(s) := a_r$ if $\frac{r}{m} =: \frac{\alpha_q}{m} \in P(B)$ else $a_r(s) := s \cdot a_r$.

Claim 1: Then ζ_s is an isotopy.

Proof: ζ_s is an immersion, for each s , since $dx/d\theta \neq 0$, where $x = \rho e^{im\theta}$.

Injectivity of $(\mathbb{C}, 0) \ni t \to (t^{m_0}, y(t)) \in (\mathbb{C}^2, 0)$ for $y(t) = \sum_{k \geq 1} a_k t^{n_k}$,

all $a_k \neq 0$. Say $t_1^{m_0} = t_2^{m_0}$, i.e. $t_2 = \epsilon^j t_1$, where $\epsilon := \eta_{m_0}$.

Then $y(t_1) \neq y(\epsilon^j t_1)$ (for a suff. small $|t_1|$) . Indeed ,

$\exists\ k_* < \infty$ s.th. $gcd(m_0, m_1, \ldots, m_{k_*}) = 1$ and $k_0 \leq k_*$ s.th. $\epsilon^{jm_{k_0}} \neq 0$ □

Claim 2: $\psi(t,s) := (t^m, \sum_{q=1}^{g} e^{s \cdot l_q} t^{\alpha_q})$, where $e^{l_q} := a_{\alpha_q}$, is an isotopy.

Proof: Suffices to verify injectivity of $\psi(t,s)$ $\forall$ fixed s , see Claim 1. □

Pf. of Lem. 4: $K(B)$ and $K(B')$ isotopic to a canonical knot B_0

(claim 1 & 2) . □

Links: Say $\Gamma = \bigcup_{1\le j\le n} B_j$. Let $t \mapsto (t^{m_j}, h_j(t))$ desing. B_j .

Lemma 5: $\exists$ a choice of h_j , $1 \le j \le n$ s.th. $\forall$ i ,j $\mathcal{O}(h_i, h_j) = \mathcal{O}(B_i, B_j)$.

Proof: Let h_i satisifying Lemma be picked for $1 \le i < n$. Pick j s.th.

$\mathcal{O}(B_j, B_n) = max\{\mathcal{O}(B_i, B_n) : 1 \le i < n\}$. Then $\exists$ h_n s.th.

$\mathcal{O}(h_j, h_n) = \mathcal{O}(B_j, B_n)$ (see page 8) $\Rightarrow \mathcal{O}(h_i, h_n) = \mathcal{O}(h_i, h_j)$

for $1 \le i \ne j < n$. Since $h_j(x^{1/m_j}) = h_i(x^{1/m_i})$ mod (x^ℓ) ,

only for $\ell \le \ell_*$, where $h_j(x^{1/m_j}) = h_n(x^{1/m_n})$ mod (x^{ℓ_*}) .

$\Rightarrow \mathcal{O}(B_i, B_n) = \mathcal{O}(B_i, B_j) \overset{ind.hyp.}{=} \mathcal{O}(h_i, h_j) = \mathcal{O}(h_i, h_n)$. □

Pf. of Thm.: Let $t \mapsto (t^{m_i}, h_{ki}(t) = \sum_r a_{r,i}(k)t^r)$, desing.

B_{ki} , $k = 0, 1$. By Lemma 5, suffices to show $\exists$ deformations $h_{si}(t)$,

$0 \leq s \leq 1$ s.th. 1) $t \mapsto (t^{m_i}, h_{si}(t) =: \sum_r a_{r,i}(s)t^r)$ desing. B_{si} and

$P(B_{si}) = P(B_{0i})$ $\forall$ s , and $\forall i$. 2) $\mathcal{O}(B_{si}, B_{sj}) = \mathcal{O}(B_{0i}, B_{0j})$ $\forall$ s .

Induction. Base case is Lem. 4. Let $m := m_{\mathbf{0}}(B_n)$.

Suppose $a_{r,i}(s)$ picked for $1 \leq i < n$ satisfying 1) and 2) and let

$a_{r,n}(s) := a_{r,j}(s)$ for $\frac{r}{m} < \frac{p}{q} := max\{\mathcal{O}(B_n, B_j)\}$ s.th. $gcd(p, q) = 1$.

For $r = \frac{p}{q}$: if $q \notin \{\frac{m}{d_i}\}_{d_i \in D(B_n)} \subset \mathbb{Z}_+ \Rightarrow \frac{p}{q} \notin P(B_n)$ and $\frac{p}{q} \in P(B_j) \Rightarrow$

$a_{r,j}(s) \neq 0$ and $a_{r,n}(0) = a_{r,n}(1) = 0$. Then let $a_{r,n}(s) := 0$.

Else if $q \in \{\frac{m}{d_i}\}_{d_i \in D(B_n)}$, take δ s.th. $|a_{r,j}(s) - a_{r,j}(0)| < |a_{r,j}(0) - a_{r,n}(0)|$

for $s \in [0, \delta]$ and $|a_{r,j}(s) - a_{r,j}(1)| < |a_{r,j}(1) - a_{r,n}(1)|$ for $s \in [1 - \delta, 1]$.

Let $a_{r,n}(s)$ be smooth and s.th. i) $a_{r,n}(s) \neq a_{r,j}(s)$ for $s \in [0, \delta]$ and

$|a_{r,n}(\delta)| > |a_{r,j}(s)|$ for $s \in [0, 1]$, ii) $a_{r,n}(s) = a_{r,n}(\delta)$ for $s \in [\delta, 1 - \delta]$,

and iii) $a_{r,n}(s) \neq a_{r,j}(s)$ for $s \in [1 - \delta, 1]$. If $\frac{r}{m} \in P(B_n)$ avoid 0 .

For $\frac{r}{m} > \frac{p}{q}$ pick a smooth $a_{r,n}(s)$, $s \in [0, 1]$, s.th. $a_{r,n}(s) \neq 0$ whenever

$\frac{r}{m} \in P(B_n)$. On Cyl_ρ for small ρ this constructs the isotopy, as required.