Introductory Exercises I (MAT477 in 2009 - 2010):

1 DeRham Thm: $\mathbf{H}^k(M) \xrightarrow{Int_k} \mathbf{H}^k(\Sigma)$

Assume that M is a manifold, $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ is an exterior differential and that M is *triangulated*, i.e. $M =$ union of simplices $\bigcup_{\sigma\in\Sigma} \sigma$. Form a 'geometric complex' via simplex $\sigma \mapsto$ its boundary $\partial\sigma$ (for both taking into account the orientation as in Stokes formula) with oriented simplices of dimension k being (by definition) a basis of vector space Σ_k ,

$$\partial_{k-1} : \ \Sigma_k \to \Sigma_{k-1} \quad , \quad \partial^*_{k-1} : \ \Sigma^*_{k-1} \to \Sigma^*_k$$

Exercise 1. Stokes' formula $\int_{\partial D} \omega = \int_D \partial\omega \ \Rightarrow$ commutativity of

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) \\ \downarrow{\scriptstyle Int_{k-1}} & & \downarrow{\scriptstyle Int_k} \\ \Sigma^*_k & \xrightarrow{\partial^*_k} & \Sigma^*_{k+1} \end{array}$$

where Int_k is the integration against the simplices of dimension k .

Exercise 2. Show that

$$\begin{array}{ccc} \ker d_k & \xrightarrow{Int_k} & \ker \partial^*_k \\ \widetilde{Int}_k \downarrow & \swarrow & \\ {}^{\ker \partial^*_k}/_{\operatorname{im} \partial^*_{k-1}} & & \end{array}$$

is well-defined and that $\ker \widetilde{Int}_k \supseteq \operatorname{im} d_{k-1}$.

Corollary: $\widetilde{Int}_k$ induces a (well-defined) map

$$\mathbf{H}^k(M) := {}^{\ker d_k}/_{\operatorname{im} d_{k-1}} \xrightarrow{\pi} \mathbf{H}^k(\Sigma) := {}^{\ker \partial^*_k}/_{\operatorname{im} \partial^*_{k-1}}$$

Elementary forms (a topic) provide an explicit right inverse of Int_k .
Exercise 3. Using elementary forms $\Rightarrow \pi$ is onto.

Acyclic (a topic): Subcomplex $\ker(Int_k) \subseteq \Omega^k$ is acyclic.
Exercise 4. Using subtopic 'Acyclic' $\Rightarrow \pi$ is injective.

2 Resultants.

Let $P(y,a) := y^p + \sum_i a_i y^{p-i}$ and $Q(y,b) = y^q + \sum_j b_j y^{q-j}$. Consider $F(y,c)|_{c=c(a,b)} := y^d + \sum_k c_k(a,b) y^{d-k} := P(y,a)\cdot Q(y,b)$, where $d := p+q$, and let resultant of $P(y,a)$ and $Q(y,b)$ (in y) be $\mathrm{res}_{P,Q}(a,b) := \det \partial c/\partial(a,b)$.

Exercises 5.

(a) For $P(y,a(\lambda)) := \prod_{1\le s\le p}(y-\lambda_s)$ and $Q(y,b(\mu)) := \prod_{1\le j\le q}(y-\mu_j)$ show that $\mathrm{res}_{P,Q}(a(\lambda),b(\mu)) = \prod_{\substack{1\le s\le p\\ 1\le j\le q}}(\lambda_s-\mu_j)$.

(b) Consider polynomials $a_i(\lambda)$ and $b_j(\mu)$ in λ and μ defined in (a) (called *elementary symmetric polynomials*). Show that $F\in \Bbbk[a_1,\ldots,a_p]$ and $F(a(\lambda))\equiv 0$ implies $F\equiv 0$. Similarly $G(a,b)\in\Bbbk[a,b]$ and $G(a(\lambda),b(\mu))\equiv 0$ implies $G(a,b)\equiv 0$.

(c) Using (b), show that for any $L(y,c) = y^l + \sum_{k=1}^{l} c_k y^{l-k}$,

$$\mathrm{res}_{P\cdot Q,L}(a,b,c) = \mathrm{res}_{P,L}(a,c)\cdot \mathrm{res}_{Q,L}(b,c)$$

(d) In the 3 exercises below when $\Bbbk\neq\mathbb{R}$ and $\Bbbk\neq\mathbb{C}$ but rather $\Bbbk$ is any field of characteristic 0 replace the rings $\Bbbk\{\cdot\}$ of convergent power series by the rings $\Bbbk[[\cdot]]$ of formal power series expansions (both with coefficients in $\Bbbk$). The exercise here is to detect in which of these 3 exercises it is essential to assume that field $\Bbbk$ is of characteristic 0 .

(e) Using the definition of $\mathrm{res}_{P,Q}(a,b)$ show that if at $\tilde c := c(\tilde a\ ,\ \tilde b)\in\Bbbk^d$ $\mathrm{res}_{P,Q}(\tilde a\ ,\ \tilde b)\neq 0$ then exist $a_i(c)$, $b_j(c)\in\Bbbk\{(c-\tilde c)\}$, $1\le i\le p$, $1\le j\le q$, such that $F(y,c))\equiv P(y,a(c))\cdot Q(y,b(c))$ in $\Bbbk\{(c-\tilde c)\}[y]$.

(f) Using (d) above show for any $F(y,c(x))\in\Bbbk\{x\}[y]$, where both x and y are single variables, such that $F(y,c(x))$ is monic in y with $c_1\equiv 0$ and some $c_{k_0}(0)\neq 0$), that whenever $\min_{1\le k\le d}(1/k)\cdot \mathrm{ord}_x\, c_k(x)$ is an integer expansion $F(y,c(x))$ is a product in $\Bbbk\{x\}[y]$ of $P(y,a(x))$ and $Q(y,b(x))$.

(g) **Puiseux Expansion.** Show for any $F(y,c(x))\in\Bbbk\{x\}[y]$ that $F(y,c(t^{d!}) = \prod_{k=1}^{d}(y-f_k(t))$ in $\Bbbk\{t\}[y]$.

Homogenization: Consider $HP(x_0,\dots,x_n) := P(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})\cdot x_0^p$, where $p=\deg P$ and $P\in \Bbbk[x_1,\dots,x_n]$. Then HP and $\mathcal{L}P(x_1,\dots,x_n) := HP(0,x_1,\dots,x_n)$ are homogenous polynomials of degree p. Consider map $j:\Bbbk^n \hookrightarrow \Bbbk\mathbb{P}^n := \{\text{lines through 0 in } \Bbbk^{n+1}\}$ defined by $j(x_1,\dots,x_n) := [1:x_1:\dots:x_n]\in \Bbbk\mathbb{P}^n$. Then $P(x)=0$ iff $HP(j(x))=0$, while $\Bbbk\mathbb{P}^n\setminus j(\Bbbk^n) = \Bbbk\mathbb{P}^{n-1} = \{[x_0:x_1:\dots:x_n]\in\Bbbk\mathbb{P}^n : x_0=0\}$.

Exercise 6. Assume $\deg P = \deg_y P$ and $\deg Q = \deg_y Q$. Then $\deg_x \operatorname{res}_{P,Q}(x) < \deg P\cdot degQ$ iff $\{(x,y)\in\mathbb{C}^2 : \mathcal{L}P(x,y)=0=\mathcal{L}Q(x,y)\} \neq \{(0,0)\}$.

3 Rings $\Bbbk[X]_a$, $\Bbbk[[X]]$ and $\Bbbk\{X\}$.

Exercise 7. Let $\Bbbk[[X_1,\dots,X_n]]$ be the ring of formal power series expansions in $X=(X_1,\dots,X_n)$ with coefficients in $\Bbbk$ (for $F\in\Bbbk[[X]]$ we write $F=\sum c_\alpha X_1^{\alpha_1}\cdots X_n^{\alpha_n}$). When $\Bbbk=\mathbb{R}$ or $\mathbb{C}$ consider also subring $\Bbbk\{X\} := \{F\in\Bbbk[[X]] : F \text{ has a positive radii of convergence }\}$, i.e. the ring of analytic near $0\in\Bbbk^n$ functions. Let $\Bbbk[X]_a$ denote the ring of quotients $\frac{P}{Q}$ of polynomials $P,Q\in\Bbbk[X]$ such that $Q(a)\neq 0$. Then for $\Bbbk=\mathbb{R}$ or $\mathbb{C}$ there are inclusions $\Bbbk[X]_0\hookrightarrow\Bbbk\{X\}\hookrightarrow\Bbbk[[X]]$. Show that for any collection of polynomials $P_1,\dots,P_s$ vanishing at $a=0\in\Bbbk^n$ it follows $\hat I\cap\Bbbk\{X\}=I^\omega$ and $\hat I\cap\Bbbk[X]_0 = I$, where ideals $\hat I$, I^ω and I are generated by these polynomials in rings $\Bbbk[[X]]$, $\Bbbk\{X\}$ and $\Bbbk[X]_0$ respectively. Conclude that

$$\Bbbk[X]_0/I\hookrightarrow\Bbbk\{X\}/I^\omega\hookrightarrow\Bbbk[[X]]/\hat I$$

are inclusions.
Advice: Consult with theorems on early pages of the book “Algebraic Geometry. I Complex Projective Varieties” by D. Mumford (a possible topic).

Exercise 8. Let $z=(z_1,\dots,z_n)$ and $\mathbb{N}:=\{0,1,2,\dots\}$. Prove that $\hat{\mathcal{O}}_n := \mathbb{C}[[z]] = \hat I\oplus\hat{\mathcal{O}}_n^{\mathcal{N}}$ and $\mathcal{O}_n^\omega := \mathbb{C}\{z\} = I^\omega\oplus(\mathcal{O}_n^\omega)^{\mathcal{N}}$, where $\mathcal{N}\subset\mathbb{N}^n$ is the *diagram* (means subset satisfying $\alpha\in\mathcal{N}$ and $\beta\in\mathbb{N}^n$ implies $(\alpha+\beta)\in\mathcal{N}$) of the *initial exponents* ($\alpha\in\ \mathbb{N}^n$) of the expansions in $\hat I$ (or, equivalently, in I^ω) and, by definition, $\hat{\mathcal{O}}_n^{\mathcal{N}}\subset\hat{\mathcal{O}}_n$ consists of all expansions $F\in\hat{\mathcal{O}}_n$ with $F=\sum c_\alpha z^\alpha$, where all $c_\alpha=0$ for $\alpha\in\mathcal{N}$, and

$(\mathcal{O}_n^\omega)^\mathcal{N} := \hat{O}_n{}^\mathcal{N} \cap \mathbb{C}\{z\}$. Conclude that $\hat{\mathcal{O}}_n$ and $\mathcal{O}_n^\omega$ are Noetherian rings.

Exercise 9. Using exercise 8 derive Weierstrass Division Theorem: Let $x = (x_1, \ldots, x_n)$, but y be a single variable. If $P(y, a(x)) = y^d + \sum_{j=1}^d a_j(x) y^{d-j}$ with $a_j \in \mathbb{C}[[x]]$ (respectively $a_j \in \mathbb{C}\{x\}$) then $\forall F \in \mathbb{C}[[x, y]]$ (respectively $\forall F \in \mathbb{C}\{x, y\}$) there exists a unique $R(x, y) = \sum_{j=1}^d r_j(x) y^{d-j} \in \mathbb{C}[[x, y]]$ such that $F = Q \cdot P + R$ and $Q \in \mathbb{C}[[x, y]]$ (respectively R and $Q \in \mathbb{C}\{x, y\}$). As a consequence prove Weierstrass Preparation Theorem: Every $f \in \mathbb{C}[[x, y]]$ (respectively in $\mathbb{C}\{x\}$) with $\operatorname{ord}_y f(0, y) = d \neq 0$ coincides (up to an invertible factor in the respective ring) with some $P(y, a(x))$.

Exercise 10. Show using Weierstrass Preparation and Division Theorems that rings $\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$ are unique factorization domains.

4 Bezout Theorem.

Exercise 11. (a) Using previous 3 exercises show that for any ideal I^ω in $\mathbb{C}\{z\}$ such that $I^\omega \cap \mathbb{C}\{z_1, \ldots, z_m\} = \{0\}$, $0 < m < n$, it follows that $0 \in \mathbb{C}^n$ is not an isolated point of the set of common zeroes of $f \in I^\omega$.

(b) Let $\{P_j\}_{j \leq n} \subset \mathbb{C}[z]$ and assume that $0 \in \mathbb{C}^n$ is an isolated point of $\{z \in \mathbb{C}^n : P_1(z) = \ldots = P_n(z) = 0\}$. Show that then $\dim_\mathbb{C} \mathbb{C}\{z\}/I^\omega < \infty$ and that $\mathbb{C}\{z\}/I^\omega \hookrightarrow \mathbb{C}[[z]]/\hat{I}$ is an isomorphism, where I^ω and $\hat{I}$ are the ideals generated by P_j's in $\mathbb{C}\{z\}$ and $\mathbb{C}[[z]]$ respectively.
Hint. Use previous 3 exercises and (a).

(c) $\forall\ f \in \mathbb{C}[[z]]$ let $(in_0 f)(z) := [t^{-\operatorname{ord}_0 f} \cdot f(t \cdot z)]|_{t=0}$. For $\{P_j\}_{j \leq 2}$ from (b) and coordinates $z = (z_1, z_2)$ such that $(in_0 P_j)(z_1, 0) \neq 0$, $j = 1, 2$, let $\operatorname{res}_{P_1, P_2}(z_2) := \operatorname{res}_{\tilde{P}_1, \tilde{P}_2}(z_2)$, where monic $\tilde{P}_j \in \mathbb{C}\{z_2\}[z_1]$ are provided by Weirstrass Preparation Theorem (with $\deg_{z_1} \tilde{P}_j = \operatorname{ord}_0 P_j$ and the same ideals generated in $\mathbb{C}\{z\}$ by $\tilde{P}_j$ and P_j , $j = 1, 2$). Show that

$$\dim_\mathbb{C} {}^{\mathbb{C}\{z_1, z_2\}}/_{(P_1, P_2)} = \operatorname{ord}_0 \operatorname{res}_{P_1, P_2}(z_2)$$

Advice: Consult regarding the properties of $\dim_\mathbb{C} {}^{\mathbb{C}\{z_1, z_2\}}/_{(P_1, P_2)}$ with theorems in the book on "Algebraic curves" by W. Fulton (a possible topic).

Exercise 12. Prove Bezout's Theorem in $\dim = 2$: Assume $\#\{(x,y) \in \mathbb{C}^2 : P(x,y) = Q(x,y) = 0\} < \infty$. Show that

$$\sum_{(a,b)\in V(P,Q)} \mathrm{mult}_{(a,b)}(P,Q) \le \deg P \cdot \deg Q \quad ,$$

where $\mathrm{mult}_{(a,b)}(P,Q) := \dim_{\mathbb{C}} \mathbb{C}[x,y]_{(a,b)}/(P,Q)$ and, moreover, $\sum_{(a,b)\in V(P,Q)} \mathrm{mult}_{(a,b)}(P,Q) = \deg P \cdot \deg Q$ iff there are no roots at ∞, i.e. $(\mathcal{L}P)(x,y) = 0 = (\mathcal{L}Q)(x,y) \Rightarrow (x,y) = (0,0)$.

5 Sard Theorem and applications.

Exercise 13. $\forall$ closed $X \subset \mathbb{R}^n$ $\exists$ $f \in C^\infty(\mathbb{R}^n\ ,\ \mathbb{R})$ such that $f(x) \ge 0\ \forall x$ and $X = \{x : f(x) = 0\}$.

Exercise 14. Assume open $U \subset \mathbb{R}^n$, $\phi \in C^\infty(U, \mathbb{R}^m)$ and closed in U $Z_\phi := \{x \in U : \mathrm{rank}\, D\phi(x) < m\}$. Let $b \in \phi(U) \setminus \phi(Z_\phi)$. Then $\phi^{-1}(b)$ is a C^∞-submanifold of U of dimension $m - n$. If $a \in \phi^{-1}(b)$ then there are C^∞-coordinate changes in $\mathbb{R}^n$ near a and in $\mathbb{R}^m$ near b such that ϕ is a linear map near a .

Exercise 15. Let M^m be a C^2-submanifold in $\mathbb{R}^n$ and $f \in C^2(M^m\ ,\ \mathbb{R})$. Let $a \in M$ and choose coordinates $(x_1, \ldots, x_m)$ on M near a . Show that the following two properties do not depend on the choice of coordinates:

- $\nabla f(a) \neq 0$.
- $\nabla f(a) = 0$ and $\det \mathrm{Hess}_f(a) \neq 0$.

Exercise 16. Assume $M = \mathrm{graph}\,\psi$, where open $U \subset \mathbb{R}^m$, $\psi : U \to \mathbb{R}^{n-m}$ and $f_{c,\theta}(x,y) = \sum_{j=1}^m c_j x_j + \sum_{j=1}^{n-m} \theta_j y_j$ and $g_{c,\theta} = f_{c,\theta}(x, \psi(x))$. Let map $\phi : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ be defined by $\phi : (x,\theta) \mapsto (\sum_{j=1}^{n-m} \theta_j \nabla\psi_j(x)\ ,\ -\theta)$. Then $(\nabla g_{c,\theta})(a) = 0$ iff $-(c,\theta) = \phi(a,\theta)$ and $\det|\mathrm{Hess}_{g_{c,\theta}}(a)| \neq 0$ iff $|D\phi(a,\theta)| \neq 0$.

Exercise 17. Using Sard's theorem, and solution of exercises 15 and 16, show that for any C^2-submanifold M of $\mathbb{R}^n$ and for "almost all" choices of $c \in \mathbb{R}^n$ the restriction to M of function $f := \sum_{j=1}^n c_j x_j$ is a Morse function, i.e. for every critical point $a \in M$ of f, $|\mathrm{Hess}_f(a)| \neq 0$.

Exercise 18. Let M^m be a compact C^2-manifold (i.e. covered by finitely many coordinate charts with C^2-transition functions). Assume that $h : M \to \mathbb{R}^n$ is a C^2-map with $\operatorname{rank} Dh(a) = m$ for all $a \in M$. Show that for "almost all" choices of $c \in \mathbb{R}^n$ function $f_c(x) := \sum_{j=1}^n c_j h_j(x)$ is a Morse function.

Exercise 19. By definition $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are real and complex projective spaces of dimension n, i.e. all $\mathbb{k}$-lines passing through 0 in $\mathbb{k}^{n+1}$, where $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ respectively, with homogenous coordinates $[z_0 : z_1 : \ldots : z_n]$. Every nondegenerate linear map $L : \mathbb{k}^{n+1} \to \mathbb{k}^{n+1}$ induces a coordinate change $[w] = h([z])$. Show that collection of functions (on $\mathbb{CP}^n$) $\mathrm{Re}\,\frac{z_j \bar{z}_k}{\sum_{0 \le s \le n} |z_s|^2}$ and $\mathrm{Im}\,\frac{z_j \bar{z}_k}{\sum_{0 \le s \le n} |z_s|^2}$, $0 \le j, k \le n$, satisfies the assumption in exercise 18.

Exercise 20. Using exercise 19, show that for any (real) C^2-submanifold $M \subset \mathbb{CP}^n$ there exists a choice of $\mathbb{C}$-homogenous coordinates $[z_0 : \ldots : z_n]$ on $\mathbb{CP}^n$ and numbers $c_j \in \mathbb{R}$ such that the restriction $f : M \to \mathbb{R}$ of function $f(x) = \sum_{0 \le j \le n} c_j \frac{|z_j|^2}{\sum_{0 \le s \le n} |z_s|^2}$ to M is a Morse function.