Resolution of Singularities in Char. 0 (Pt. 2)

inv_I and proof of global desingularization

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Enticement

Want 'geometric desing.' of $X \subset M$, i.e. $\phi : M' \to M$ with

$\mathrm{Sing}(X) = \mathrm{Sing}(\phi)$ and 'lifted' $X' := cl(\phi^{-1}(X \setminus \mathrm{Sing}X))$ nonsing.

This talk: algebraic desing., i.e. $\det(J_\phi) \cdot (I_X \circ \phi)$ locally monomial

and ϕ a composite of blowings-up with nonsing. centres

Remark. (Process of) algebraic desing. $\implies$ (Weak) geometric desing.:

Stop blowing up when inv_I^{ext} constant on resp. $X' \subset M'$

inv_I^{ext} constant on $X' \implies X'$ nonsingular

Year 0. Effect of blowing-up revisited: 'maximal contact' N

$N_0 := M \supset U_0 \ni a\,;$ $\quad I_X = (f)\,;$ $\quad \mathcal{F}(a) := \{(f, \mu_a(f))\}\,;$ $\quad \nu_1(a) := \mu_a(f) =: \mu_1(a)$

(in gen., $\mathcal{F}(a) := \{(f, \mu_f)\}$ $\qquad$ e.g. $I_X := (f_1, \dots, f_k)$, $\quad \mu_{f_i} := \min_j \mu_a(f_j))$

$\mathcal{F}(a)$ is chosen s. th. $\quad \nu_1(\cdot)$ at a $\sim \mathcal{F}(a)$, i.e.

$$\{x : \nu_1(x) = \nu_1(a)\} =: S_{\nu_1}(a) \overset{\text{'stably'}}{=} S_{\mathcal{F}(a)} := \{x : \mu_x(f) \geq \mu_f\ , (f, \mu_f) \in \mathcal{F}(a)\}$$

$$\exists f_* \in \mathcal{F}(a) \text{ s.th. } \mu_a(f_*) = \mu_{f_*} =: d \implies \frac{\partial^d f_*}{\partial {x_n}^d}(a) \neq 0 \text{ (after lin. tran)}$$

Passage to codim 1, same as Slide 6 of Talk 1 (in year 0, $\mathcal{F}_0(a) := \mathcal{F}(a)$) :

$$N_1 := \left\{ \frac{\partial^{d-1} f_*}{\partial {x_n}^{d-1}} = 0 \right\} \ ; \quad \mathcal{G}_1(a) := \left\{ \left(c_{f,q} := \left. \frac{\partial^q f}{\partial {x_n}^q} \right|_{N_1} , \mu_f - q \right) : \begin{array}{l} 0 \leq q < \mu_f\ , \\ (f\ ,\ \mu_f) \in \mathcal{F}_0(a) \end{array} \right\}$$

Ex. In year 0, $f = x_3^2 - x_1^2 x_2^3$ at $a = 0\,;$ $\qquad N_1 = \{x_3 = 0\}\,;$ $\qquad \mathcal{G}_1(0) = \{(x_1^2 x_2^3, 2)\}$

Year k. Set-up

$$
\begin{array}{ccccccccccccc}
a =: a_k & \mapsto & a_{k-1} & \mapsto & \cdots & \mapsto & a_i & \mapsto & a_{i-1} & \mapsto & \cdots & \mapsto & a_0 \\
M_k & \xrightarrow{\sigma_k} & M_{k-1} & \xrightarrow{\sigma_{k-1}} & \cdots & \xrightarrow{\sigma_{i+1}} & M_i & \xrightarrow{\sigma_i} & M_{i-1} & \xrightarrow{\sigma_{i-1}} & \cdots & \xrightarrow{\sigma_1} & M_0 := U_0 \\
\nu_1(a_k) & = & \nu_1(a_{k-1}) & = & \cdots & = & \nu_1(a_i) & < & \nu_1(a_{i-1}) & & E_{old\ for\ \nu_1} & := & \{H_j\}_{j\le i}
\end{array}
$$

(σ_i bl.-up with centre C_{i-1}) $\qquad H_i := \sigma_i^{-1}(C_{i-1})$ (index 'lifted' to M_k)

$$
\left.\begin{array}{r}
\mathcal{F}(a_k) = \mathcal{F}(a_{k-1})' := \{(f' := y_{H_k}^{-\mu_f}(f\circ\sigma_k)\ ,\ \mu_{f'} := \mu_f)\} \text{ for } (f,\mu_f) \in \mathcal{F}(a_{k-1}) \\
N_1(a_k) = N_1(a_{k-1})' \\
\mathcal{G}_1(a_k) = \mathcal{G}_1(a_{k-1})'
\end{array}\right\} k \ge j > i
$$

$$
E^1(a_k) := \{H \in E_{old\ for\ \nu_1} : a_k \in H\} \subset E(a_k), \qquad s_1(a_k) := \#E^1(a_k)
$$

$$
\mathcal{E}_1(a_k) := E(a_k) \setminus E^1(a_k)
$$

Year k. Set-up cont'd. $\mathrm{inv}_{1.5}(a_k) := (\nu_1(a_k), s_1(a_k); \nu_2(a_k))$

$$\mu_2(a_k) := \min_{(g\,,\,\mu_g)\in\mathcal{G}_1(a_{k-1})} \left(\frac{\mu_{a_{k-1}}(g)}{\mu_g}\right) =: \nu_2(a_k) + \sum_{H\in\mathcal{E}_1(a_k)} \mu_{2H}(a_k)\ , \text{ where}$$

$$\mu_{2H}(a_k) := \min_{(g\,,\,\mu_g)\in\mathcal{G}_1(a_{k-1})} \frac{\text{order to which } y_H \text{ factors from } g}{\mu_g}\ , \quad I_H = (y_H)$$

$$\mathcal{F}_1(a_k) := (*) \cup \{(D_1(a_k)\,,\, 1-\nu_2(a_k))\}\ , \text{ where } D_1(a_k) := \prod_{H\in\mathcal{E}_1(a_k)} y_H^{\mu_{2H}(a_k)}$$

$$\text{and } (*) := \{(D_1(a_k)^{-\mu_g}\cdot g, \nu_2(a_k)\cdot\mu_g) : (g,\mu_g)\in\mathcal{G}_1(a_k)\}$$

Consequently $\mathrm{inv}_{1.5}(\cdot)$ at $a_k \sim \mathcal{F}_1(a_k)$

$$E^2(a_k) := \{H \in E_{old\ for\ \mathrm{inv}_{1.5}} : a_k \in H\} \subset \mathcal{E}_1(a_k)\ , \qquad s_2(a_k) := \#E^2(a_k)$$

$$\mathcal{E}_2(a_k) := \mathcal{E}_1(a_k) \setminus E^2(a_k)$$

Year 0. ν_r, E^r, s_r, $\mathcal{E}_r$ at a defined recursively

Ex. In year 0, $\nu_2(a) = \mu_2(a) = \frac{5}{2}$; $\mathcal{F}_1(a) := \{(x_1^2 x_2^3, 5)\} \sim \{(x_1, 1), (x_2, 1)\}$

To N_2 : $N_2(a) = N_1(a) \cap \{x_2 = 0\}$; $\mathcal{G}_2(a) = \{(x_1, 1)\}$; $\mathcal{F}_2(a) = \mathcal{G}_2(a)$

$$\mathrm{inv}_{1.5}(a) = (2, 0; 5/2) ; \quad \mathrm{inv}_2(a) = (\mathrm{inv}_{1.5}(a), 0)$$

To N_3 : $N_3(a) = N_2(a) \cap \{x_1 = 0\} = \{a\}$; $\mathcal{G}_3(a) = \emptyset = \mathcal{F}_3(a)$

$$\mathrm{inv}_l(a) = (\mathrm{inv}_2(a); 1, 0; \infty) .$$

Year k : Last $\nu_{r+1} = \infty$ or 0. If ∞, bl.-up with centre $S_{\mathrm{inv}_l}(a) = N_r$

Ex. In year 0, $\sigma_1 : M_1 \to M_0$ with $C_0 = \{a\}$

Year 1. 'New' for $\mathrm{inv}_{0.5}$ and 'old' for $\mathrm{inv}_{1.5}$ exc. div.

Ex. In year 1, $M_1 := Bl_{C_0}(U_0) \supset U_i$; $\quad y_{exc} = y_i$, $\quad H_1 = \{y_{exc} = 0\}$

$b := 0 \in U_1$; $\quad \sigma_1|_{U_1} : \ x_1 = y_1 , \quad x_2 = y_1 y_2 , \quad x_3 = y_1 y_3$

$f_1 := f' = y_3^2 - y_1^3 y_2^3 \implies \nu_1(b) = \nu_1(a) = 2 \implies H_1 \in \mathcal{E}_1(b)$; $\quad \mathrm{inv}_1(b) = (2, 0)$

Year k : 'Account for' exc. hypersurfaces (similar to Slide 15 of Talk 1)

$H \in \mathcal{E}_1(a_k)$ ('new for $\mathrm{inv}_{0.5}$') : In def. of $\mathcal{F}_1(a_k)$

$H \in E^2(a_k)$ ('old for $\mathrm{inv}_{1.5}$') : $\mathcal{F}_1^s(a_k) := \mathcal{F}_1(a_k) \cup \Big(\bigcup_{H \in E^2(a_k)} \{(y_H, 1)\}\Big)$

$$\mathrm{inv}_2(\cdot) \text{ at } a_k \sim \mathcal{F}_1^s(a_k)$$

Years 1 and 2. Case $\mathrm{inv}_I(a) = (\dots; 0)$

Ex. In year 1, $N_1(b) = \{y_3 = 0\}$; $\mathcal{G}_1(b) = \{(y_1^3 y_2^3\ , 2)\}$; $\nu_2(b) = \frac{3}{2} \implies H_1 \in E^2(b)$

$D_1(b) = y_1^{3/2}$; $\mathcal{F}_1(b) = \{(y_2^3\ , 2 \cdot \frac{3}{2})\} \sim \{(y_2\ , 1)\}$; $\mathcal{F}_1^s(b) \sim \{(y_2, 1), (y_1, 1)\}$

To N_2 : $N_2(b) = N_1(b) \cap \{y_2 = 0\}$; $\mathcal{G}_2(b) = \{(y_1, 1)\} = \mathcal{F}_2(b) = \mathcal{F}_2^s(b)$

$$\text{Recursion: } \xrightarrow{\text{increase codim } (*-1)} N_*(a_k) \to \mathcal{G}_*(a_k) \to D_*(a_k) \to \mathcal{F}_*(a_k) \to \mathcal{F}_*^s(a_k) \xrightarrow{\text{increase codim } *}$$

Ex. In year 1, $\mathrm{inv}_I(b) = (2, 0; \frac{3}{2}, 1; 1, 0; \infty)$; $C_1 = \{b\}$

Year k : $\mathrm{inv}_I(a_k) = (\dots, 0) \overset{\text{Talk 1 Slide 9}}{\Longrightarrow} S_{\mathrm{inv}_I}(a_k) = \cup_J Z_J$, where Z_J is component of $S_{\mathrm{inv}_I}(a_k)$,

$J := \{H \in E(a_k) : H \supset Z_J\}$, $Z_J = S_{\mathrm{inv}_I}(a_k) \cap \bigcap_{H \in J} H$

Year 2. Case $\text{inv}_I(a) = (\ldots; 0)$ continued

Ex. In year 2, $0 =: c \in U_{12}$; $\sigma_2|_{U_{12}} : y_1 = z_2 z_1$, $y_2 = z_2$, $y_3 = z_2 z_3$; $f_2 = z_3^2 - z_1^3 z_2^4$

$E(c) = \mathcal{E}_1(c) = \{H_1, H_2\}$, $H_i = \{z_i = 0\}$; $N_1(c) = \{z_3 = 0\}$; $\mathcal{G}_1(c) = \{(z_1^3 z_2^4, 2)\}$

$$\text{inv}_I(c) = (2, 0; 0) \text{ because } D_1(c) = z_1^{\frac{3}{2}} z_2^2 \implies \mathcal{F}_1^s(c) = \{(D_1(c), 1)\}$$

$$S_{\text{inv}_I}(c) = (S_{\text{inv}_I}(c) \cap H_1) \bigcup (S_{\text{inv}_I}(c) \cap H_2)$$

Year k : Subsets of $E_k := \{H_j\}_{1 \le j \le k}$ can be totally ordered

Take $C_k := Z_{J(a_k)}$ for $J(a_k) := \max\{J : Z_J \text{ is a component of } S_{\text{inv}_I}(a_k)\}$

Ex. In year 2, $C_2 := S_{\text{inv}_I}(c) \cap H_1$

Effect of blowing-up (Slides 6–8 & 10–11 of Talk 1) shows

Year k : $\mathrm{inv}_I(a_k) = (\dots; v_r(a_k))$; $D_{r-1}(a_k) = \prod_{H \in \mathcal{E}_{r-1}(a_k)} y_H^{\mu_{rH}}$

Case 1: $v_r(a_k) = \infty \implies S_{\mathrm{inv}_I}(a_k) = N_r(a_k) \implies \mathrm{inv}_I$ decreases ('above' a_k)

Case 2: $v_r(a_k) = 0 \implies S_{\mathrm{inv}_I}(a_k) = \cup_J Z_J \implies \mathrm{inv}_I$ decreases after at most finitely many blowings-up 'controlled' by the numerator of $\sum_{H \in \mathcal{E}_{r-1}(a_k)} \mu_{rH}$

(Just as in Talk 1 : decrease of $|\Omega|$ with $r = 2$)

In each year: pass to higher codim. until Case 1 or Case 2 holds

Proof of invariance — main technique in an example

Ex. $M := U := \mathbb{A}^2$; $f := f_0 := x_2^p - x_1^q$, $p \leq q$; $0 =: a \in U$

Show $\mu_2(a) = \frac{q}{p}$ is invariant

$U_0 := U \times \mathbb{A}^1$; $\gamma_0 := \{a\} \times \mathbb{A}^1$. For $j \geq 1$, $\gamma_j :=$ 'lifting' of γ_{j-1} to $Bl_{C_{j-1}}(U_{j-1})$, where U_j is coord. chart containing $\gamma_j(0) =: C_j$

In U_1 : $x_1 = y_1 z$, $x_2 = y_2 z$, $z = z$ $\implies$ $f_1 = y_2^p - y_1^q z^{(q-p)}$

After k such blowings-up, $f_k = y_2^p - y_1^q z^{(q-p)k}$ in U_k

$S_{(f_k\ ,p)} = \{y := (y_1, y_2, z) : y_2 = 0\ , \quad \mu_y(y_1^q z^{(q-p)k}) \geq p\}$

Proof of invariance — continuation

$H_k :=$ 'Last exc. hypersurface' $= \{z = 0\}$

'Invariant' question: Is $H_k \cap S_{(f_k\ ,p)}$ nonsing. of codim. 1 in H_k?

Yes $\iff (q-p)k \geq p \iff (\frac{q}{p}-1)k \geq 1$

If 'yes', $H_k \cap S_{(f_k\ ,p)}$ coincides with $\{y_2 = 0\ ,\ z = 0\} =: C_{k,0}$; $U_{k,0} := U_k$

In $U_{k,1}$, $y_1 = v_1,\ y_2 = v_2 z,\ z = z \implies f_{k,1} = v_2^p - v_1^q z^{(q-p)k-p}$

Proof of invariance — conclusion

After s such blowings-up, $f_{k,s} = v_2^p - v_1^q z^{(q-p)k-ps}$ in $U_{k,s}$

$$S_{(f_{k,s}\ ,p)} = \{v := (v_1, v_2, z) : v_2 = 0\ , \quad \mu_v(v_1^q z^{(q-p)k-ps}) \geq p\}$$

$H_{k,s} = \{z = 0\}$. Ask the same 'invariant question':

Yes $\iff (q-p)k - ps \geq p \iff (\frac{q}{p} - 1)k - s \geq 1$

i.e. $\frac{q}{p} = 1 + \sup\limits_{k,s \text{ with 'Yes'}} \frac{s+1}{k}$ Done

Globalization via $\text{inv}_I^{ext}(a_k) := (\text{inv}_I(a_k)\,,\ J(a_k))$

General fact: $S_{\text{inv}_I^{ext}(a_k)} = Z_{J(a_k)}$, i.e. is nonsingular

(We used this to show "algebraic desing. $\implies$ (weak) geometric desing.")

In year k, M_k Compact $\implies$ inv_I^{ext} takes on maximum value $\text{inv}_I^{ext}(M_k)$

Choose as 'global' centre $C_k := S_{\text{inv}_I^{ext}(M_k)} := \{y \in M_k : \text{inv}_I^{ext}(y) = \text{inv}_I^{ext}(M_k)\}$

Nonsingular locally $\implies$ nonsingular

Algebraic desing.: $\det(J_\phi) \cdot (I_X \circ \phi)$ locally monomial in y_{H_j}

(i). From Talk 1: By construction, $\forall j \quad C_j$ has 'normal crossings' with exc. divisors E_j

(ii). After fin. many blowings-up $:= \phi$, $\max_{M'} \mathrm{inv}_{0.5} = 0 \iff I_{X'}$ loc. gen. by invertible f

(i) and (ii) $\implies (I_X \circ \phi)$ loc. monomial in y_{H_j} (up to an invertible factor)

Say $\phi := \sigma_1 \circ \cdots \circ \sigma_q$, where σ_j is bl.-up with centre $C_{j-1} := \{x_1 = \cdots = x_{m_j} = 0\}$

$$\sigma_1|_{U_1} : x_1 = y_H \,, \quad x_i = y_i y_H \; (1 < i \le m) \,, \quad x_i = y_i \; (i > m) \implies \det(J_{\sigma_1}) = y_H^{m-1}$$

By Chain Rule and multiplicativity of det , $\det(J_\phi) = \prod_{1 \le j \le q} \det(J_{\sigma_j}) = \prod_{1 \le j \le q} y_{H_j}^{m_j - 1}$

with (i) $\implies$ Done