An Introduction to Degree of Smooth Maps

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Motivation

$f : M \to N$, e.g $M = N = S^n$, is C^∞.

What is $\# f^{-1}(y)$ for a regular value y?

Counted properly, it is a constant called top. deg. of f.

Example: $P \in \mathbb{C}[z] \Rightarrow$ top. deg. $P = Deg(P)$.

Notion of degree due to Brouwer.

Assumptions

- $M, N - C^\infty$ manifolds, no boundary, $dim(M) = dim(N)$.
- M is compact or f is proper; N is connected.
- $\forall\, f : M \to N$ are C^∞

$$Cr(f) := \{x \in M : df_x \text{ not onto}\}$$

$y \notin f(Cr(f))$ called regular value (reg. val.).

Note: y reg. val. $\Rightarrow df_x : TM_x \to TN_y$ is isom. $\forall\, x \in f^{-1}(y)$.

Requirement: M and N oriented

Fact – Complex manifolds are orientable:

$\mathbb{C}$-linear $T : \mathbb{C}^n \to \mathbb{C}^n$, then for $T_{\mathbb{R}} := T$ as $\mathbb{R}$-linear map,

$$det(T_{\mathbb{R}}) = |det(T)|^2$$

Aside: For M and N not oriented, top. deg. of $f \in \mathbb{Z}/2\mathbb{Z} \ldots$.

Def: $f, g : X \to Y$ are homotopic if $\exists\, F : X \times [0,1] \to Y$ s.th.

$F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Homotopy ($f \sim g$) is an equivalence relation.

Homotopy and Isotopy

Def: Diffeo. $f, g : X \to Y$ are isotopic ($f \simeq g$) if $\exists$ homotopy F

from f to g, s.th. $x \to F(x,t)$ are diffeo $\forall\ t \in [0,1]$.

Lemma (Milnor): $\forall\ y,\ z \in N$, $\exists$ diffeo. $h \simeq id : N \to N$ s.th. $h(y) = z$.

E.g. for S^n, isotopy constructed via rotations.

Topological Degree

$f : M \to N$

$$\mathbf{deg(f, y)} := \sum_{x \in f^{-1}(y)} sign(det(df_x)) \quad \forall \text{ reg. val. } y$$

Note: by Sard, $deg(f, y)$ is defined almost everywhere.

$\mathbf{deg(f, y)} < \infty$: Note that $f^{-1}(y)$ is compact.

$x \in f^{-1}(y) \Rightarrow df_x$ isomorphism $\Rightarrow \exists\, U \ni x$ s.th. $f|_U$ is $1:1$.

$\therefore \{x\}$ open in $f^{-1}(y) \Rightarrow f^{-1}(y)$ finite.

$def(f, y)$ is locally constant

Suppose $f^{-1}(y) = \{x_1, \ldots, x_k\}$.

By I.F.T, $\exists$ disjoint $U_i \ni x_i$ s.t $f|_{U_i}$ is diffeo. onto $V_i \ni y$.

$$\Rightarrow \# f^{-1}(y') = k,\ \forall\ y' \in V = \cap_{i=1}^{k} V_i - f(M - \cup_{i=1}^{k} U_i)$$

$f|_{U_i}$ diffeo. $\Rightarrow sign(df_x)$ const. on $U_i \Rightarrow deg(f, y)$ const. on V.

Fundamental Theorems

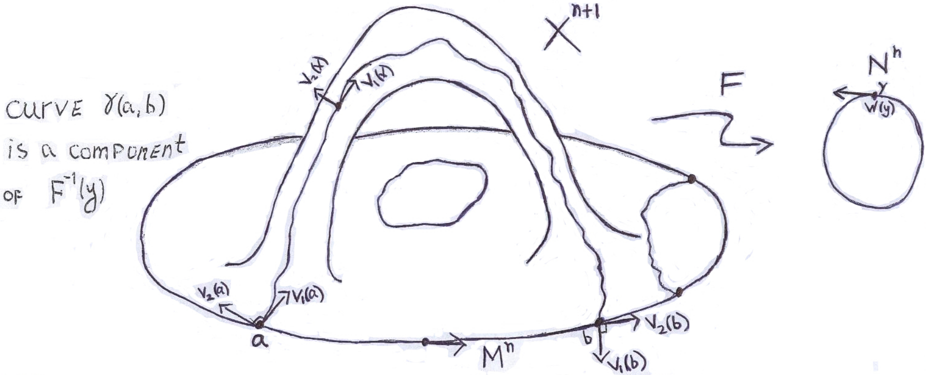
Theorem A (Well-definedness): $deg(f, y)$ doesn't depend on reg. val. y.

$$\mathbf{deg(f)} := \mathbf{deg(f, y)} \ \forall \text{ reg. val. } \mathbf{y}$$

Theorem B (Homotopy invariance): If $f \sim g$ then $deg(f) = deg(g)$.

Auxiliary Lemma: Let $M = \partial X$ with X compact, oriented and M oriented as the boundary of X . If $f : M \to N$ extends to $F : X \to N$ then $deg(f, y) = 0 \ \forall$ reg. val. y.

Proof:



$TX_x = < v_1, \ldots, v_{n+1} > \leftarrow$ means span

$T\gamma(a,b)_x = < v_1 >$

$$dF_x : \begin{array}{l} v_1 \mapsto 0 \\ < v_2, \ldots, v_{n+1} > \overset{onto}{\mapsto} TN_y \end{array}$$

$$\boxed{sign(df_a) + sign(df_b) = 0}$$

Suppose $f \sim g$ via $F : M \times [0,1] \to N$.

Lemma: $deg(f,y) = deg(g,y)$ $\forall$ common reg. val. y.

Proof : Orient $M \times [0,1]$ as product manifold.

$$\partial(M \times [0,1]) = (1 \times M) - (0 \times M)$$

$$F|^{-1}_{\partial(M\times[0,1])}(y) = (1 \times g^{-1}(y)) \sqcup (0 \times f^{-1}(y))$$

$$(1,x) \in 1 \times g^{-1}(y) \Rightarrow sign(dF_{(1,x)}) = sign(dg_x)$$

$$(0,x) \in 0 \times f^{-1}(y) \Rightarrow sign(dF_{(0,x)}) = -sign(df_x)$$

$deg(F|\partial(M \times [0,1]), y) = deg(g,y) - deg(f,y) = 0$ (Aux. Lem.).

Proof of Fundamental Theorems

For $y, z \notin f(Cr(f))$ $\exists\, h \simeq id : N \longrightarrow N$ with $h(y) = z$.

$$\begin{aligned} deg(h \circ f, h(y)) &= \sum_{x \in f^{-1}(y)} sign(d(h \circ f)_x) \\ &= \sum_{x \in f^{-1}(y)} sign(dh_y) sign(df_x) \\ &= \sum_{x \in f^{-1}(y)} sign(df_x) = deg(f, y) \end{aligned}$$

As $f \sim h \circ f \Rightarrow deg(h \circ f, h(y)) = deg(f, z)$.

Examples

- $x \mapsto c \in N$ has degree 0.
- $id : M \to M$ has degree 1.
- If $f : M \to N,\ g : N \to X$, then $deg(g \circ f) = deg(g)deg(f)$.
- The reflection $r_i : S^n \to S^n$ given by $r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$ has degree -1.
- $a(x) = -x : S^n \to S^n$ is the composition $r_1 \circ \ldots \circ r_{n+1}(x)$

$$\therefore deg(a(x)) = (-1)^{n+1} \Rightarrow a(x) \not\sim id_{S^n}$$

Degree on Complex Manifolds (Towards Application 1)

$f : M \to N$ holomorphic between complex manifolds.

Theorem: $deg(f) = \# f^{-1}(y)$ for any regular value y!!

Proof : $x \in f^{-1}(y)$ for a reg. val. $y \Rightarrow df_x$ as $\mathbb{R}$-linear map satisfies

$$det(df_x)_{\mathbb{R}} = |det(df_x)|^2 \geq 0$$

Thus $sign(df_x) = 1$ and $deg(f) = deg(f, y) = \# f^{-1}(y)$.

Mumford's Lemma

Lemma (Mumford): Let $f : X \to Y$ be continuous between a locally compact Hausdorff space X and a metric space Y.

If $f^{-1}(y)$ is compact for $y \in Y$ then $\exists$ open sets $U \supseteq f^{-1}(y)$ and $V \ni y$ such that $f(U) \subseteq V$ and $f|_U : U \to V$ is proper.

Proof : $\exists X_0 \supseteq f^{-1}(y)$ open s.th. $\bar{X}_0$ is compact.

For open $B \ni y$, *res* $f : \bar{X}_0 \cap f^{-1}(B) \to B$ is proper.

If *res* $f : X_0 \cap f^{-1}(B) \to B$ is not proper then

$$X_0 \cap f^{-1}(B) \subset \bar{X}_0 \cap f^{-1}(B)$$

If this holds $\forall$ open $B_\alpha \ni y$ then $\exists$ infinitely many distinct

$x_\alpha \in \bar{X}_0 \setminus X_0$ with $f(x_\alpha) \in B_\alpha$.

$B_\alpha \downarrow y \quad \Rightarrow \quad f(x_\alpha) \to y$

The x_α's have a limit point $x_\infty \in \bar{X}_0 \setminus X_0$.

But by continuity, $f(x_\infty) = y \Rightarrow x_\infty \in f^{-1}(y) \subseteq X_0$.

Multiplicity (Application 1)

$P : \mathbb{C}^n \to \mathbb{C}^n$ polynomial map with $a \in P^{-1}(0)$ isolated.

By Mumford's lemma, $\exists\ U \ni a$ and $V \ni 0$ s.th. $P|_U : U \to V$

is proper and $(P|_U)^{-1}(0) = a$.

Note $deg(P|_U) = \#P|_U^{-1}(y) \quad \forall$ reg. val. y. Recall that

$$a \text{ isolated zero} \Leftrightarrow \dim \mathbb{C}[\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n}]_\mathrm{a}/(\mathrm{P})_\mathrm{a} < \infty$$

$deg(P|_U) = dim$ above and one defines multiplicity of P at a

$$\mu_{\mathbf{P},\mathbf{a}} := deg(P|_U) = \dim \mathbb{C}[\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n}]_\mathrm{a}/(\mathrm{P})_\mathrm{a}$$

Brouwer Fixed Point Theorem (Application 2)

Any continuous $f : D^{n+1} \to D^{n+1}$ *has a fixed point.*

If C^∞ f has no fixed points then $\forall x \in D^{n+1}$, let $g(x) \in S^n$ be

point lying closer to x on the line segment joining x to $f(x)$.

$$g(x) = x + tu,\ u = \frac{f(x) - x}{||f(x) - x||},\ t = -x \cdot u + \sqrt{1 - ||x||^2 + (x \cdot u)^2}$$

$g : D^{n+1} \to S^n$ is smooth retraction of D^{n+1} onto S^n.

$g(0) \sim id_{S^n}$ via $F(x, t) = g(tx)\colon S^n \times [0, 1] \to S^n$. Not possible.

$f \in C^0 \Rightarrow \exists$ uniform C^∞ approximation of f via $P : D^{n+1} \to \mathbb{R}^{n+1}$

$$||f - P||_{sup} < \epsilon \Rightarrow ||P||_{sup} < 1 + \epsilon$$

Then $Q(x) := \frac{P(x)}{1+\epsilon}$ is C^∞ from $D^{n+1} \to D^{n+1}$ with

$$||f - Q||_{sup} < 2\epsilon$$

If min of $||f(x) - x||$ on D^{n+1} is $m > 0$ then min of

$||Q(x) - x|| \geq m - 2\epsilon > 0$ for small ϵ.

This is not possible by the first case.

Smooth Hairy Ball Theorem (Application 3)

A smooth tangent vector field $v : S^n \to \mathbb{R}^n$ is map satisfying

$$v(x) \cdot x = 0 \quad \forall\, x \in S^n$$

If v is a smooth tangent vector field on S^n with $v(x) \neq 0$ on S^n

then normalize i.e. $v(x) \in S^n$.

v defines homotopy $F : S^n \times [0, \pi] \to S^n$ via

$$F(x,t) = x cos(t) + v(x) sin(t);\ F(x,0) = x,\ F(x,\pi) = -x$$

$deg(F(x,\pi)) = (-1)^{n+1}$ and $deg(F(x,0)) = 1 \Rightarrow n$ is odd.

$\Rightarrow \nexists$ smooth non-vanishing tangent vector field on S^{2n}.