Sard's theorem. Proof, applications.

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Existence of Morse functions (in R^n) (Lemma)

If $f : R^n \to R$, $a = (a_1, ..., a_n) \in R^n \Rightarrow$

$measure(\{a \in R^n | g = f - a_1 x_1 - ... - a_n x_n \text{ is not Morse}\}) = 0.$

Proof: Apply Sard to $Df \Rightarrow$

$mes.(Q = \{q | Df(p) = q, det(Hess(g(p))) = 0\}) = 0.$

Let $a \notin Q$ and $g = f - a_1 x_1 - ... - a_n x_n$ is not Morse $\Rightarrow$

$\exists p$, s.t. $\text{rank}(Dg(p) = [\frac{\partial f(p)}{\partial x_1} - a_1 ... \frac{\partial f(p)}{\partial x_n} - a_n]) = 0,$

$det(Hess(g(p))) = 0 \Rightarrow Dg(p) = 0 \Rightarrow \frac{\partial f(p)}{\partial x_i} = a_i \Rightarrow$

$Df(p) = a \Rightarrow a \in Q \Rightarrow$ cont. $\Rightarrow$ if $a \notin Q \Rightarrow g$-Morse

Theorem (Sard)

U - open $\subseteq R^n$, f : U$\rightarrow R^p$ is C^∞, $p \geq 1$ C:=$\{y \in U : rank(Df_y) < p\}$
$\Rightarrow measure(f(C)) = 0.$ $/m(...) := measure(...)$

Proof: Base: n=0. Assume for n-1.

$C_1 := \{y \in U : \frac{\partial f_j}{\partial x_i}(y) = 0 \ \forall i, j\}.$

$C_i := \{y \in U : \frac{\partial^k f_j}{\partial x_{s_1} ... \partial x_{s_k}}(y) = 0 \ \forall k \leq i\} \supset C_{i+1...}$. $\forall i$, C_i- closed.

Step 1

$measure(f(C - C_1)) = 0$

If $p \geqslant 2$, $\bar{x} \in C - C_1 \Rightarrow \exists\ \frac{\partial f_i}{\partial x_j}$, say $\frac{\partial f_1}{\partial x_1}\ :\ \frac{\partial f_1}{\partial x_1}(\bar{x}) \neq 0$.

Def. $h : U \to R^n$, $h(x) := (f_1(x), x_2, ..., x_n)$.

$det(Dh_x) \neq 0 \Rightarrow \exists V \ni \bar{x}, V'\ :\ h : V \to V' \subseteq R^n$-diffeom.

$g := f \circ h^{-1} : V' \to R^p \Rightarrow C_g := \{y \in V'\ :\ \text{rank } Dg_y < p\} =$

$h(V \cap C) \Rightarrow g(C_g) = g(h(V \cap C)) = f \circ h^{-1}(h(V \cap C)) = f(V \cap C)$.

Fix t, $\forall (t, x_2, ..., x_n) \in V'$, $g((t, x_2, ..., x_n)) \in \{t\} \times R^{p-1} \subset R^p$.

Def. $g^t : (\{t\} \times R^{n-1}) \cap V' \to \{t\} \times R^{p-1}$. Then:

$$\frac{\partial g_i}{\partial x_j} = \begin{pmatrix} 1 & 0 \\ * & (\frac{\partial g_i^t}{\partial x_j}) \end{pmatrix}$$

$x \in \{t\} \times R^{n-1}$ is crit. for g^t iff x is crit. for g.

By ind. hyp. $m(g(C_g^t := \{y \in V' : \text{rank } Dg_y^t < p\})) = 0$

in $\{t\} \times R^{p-1} \Rightarrow m(g(C_g) \cap \{t\} \times R^{p-1}) = 0$

$\Rightarrow$ by Fubbini $m(g(C_g) = f(V \cap C)) = 0$.

We just used standard Theorem (Fubbini)

If $A \subset R^p = R^{p-1} \times R$ is measurable and

$measure((A \cap (constant) \times R^{p-1})) = 0$ in $R^{p-1} \Rightarrow$

$measure(A) = 0$ in R^p.

Step 2

$measure(f(C_k - C_{k+1})) = 0$

Let $\bar{x} \in C_k - C_{k+1} \Rightarrow \exists \frac{\partial^{k+1} f_j}{\partial x_{s_1} \partial x_{s_2} ... \partial x_{s_{k+1}}}(\bar{x}) \neq 0 \Rightarrow$

$w(x) := \frac{\partial^k f_j}{\partial x_{s_2} \partial x_{s_3} ... \partial x_{s_{k+1}}}(\bar{x}) = 0$. Let $s_1 = 1$.

$h(x) := (w(x), x_2, ..., x_n) : U \to R^n$ diffeom. for $V \ni \bar{x}$, $V' \Rightarrow$

$h(C_k \cap V) \subseteq 0 \times R^{n-1}$.

$g := f \circ h^{-1} : V' \to R^p$ and $\bar{g} : (0 \times R^{n-1}) \cap V' \to R^p$.

$\forall\ x \in h(C_k \cap V)$, $x \in \{y \in V' : \text{rank } Dg_y < n\}$.

By ind. hyp., $m(\bar{g}(h(C_k \cap V)) = f(C_k \cap V)) = 0$.

Step 3

$measure(f(C_k)) = 0$ if $k > \frac{n}{p} - 1$

Let $I^n \in U$ be a cube with edge δ. Let $k > \frac{n}{p} - 1$.

By Taylor $f(x+h) = f(x) + R(x,h)$. I^n - bounded $\Rightarrow$

$||R(x,h)|| \leq const.||h||^{k+1}$, $x \in C_k \cap I^n, x+h \in I^n$, c-constant.

Subdivide I^n into r^n cubes with edge $\frac{\delta}{r}$, let $x \in I_1 \Rightarrow$

$\forall y \in I_1$, $y = x + h$; $||h|| \leq \sqrt{n}\frac{\delta}{r}$, c := constant.

$||R(x,h)|| \leq c||h||^{k+1} \Rightarrow f(I_i) \subseteq$ cube of edge

$\frac{2c(\sqrt{n}\delta)^{k+1}}{r^{k+1}}$ centered at $f(x) \Rightarrow$

$V(f(C_k \cap I^n)) \leq r^n(\frac{2c(\sqrt{n}\delta)^{k+1}}{r^{k+1}})^p = (2c(\sqrt{n}\delta)^{k+1})^p r^{n-(k+1)p} \Rightarrow.$

If $\frac{n}{p} < k+1 \Rightarrow V \to 0$ as $r \to \infty \Rightarrow m(f(C_k \cap I^n)) = 0.$

SARD THEOREM PROVED.

Thm. Sard $\Rightarrow \exists$ Morse fun. on compact manifold

Let $\tilde{f}_i : R^n \to R$, s.t. $span\{d\tilde{f}_i\} = T_p^* R^n \ \forall p \in M$ - k-dim.,

Let $f_i = \tilde{f}_i|_M \Rightarrow span\{df_1, ..., df_n\} = T_p^* M$.

Thm: Sard $\Rightarrow measure(\{a \in R^n | a_1 f_1 + ... + a_n f_n$ is not Morse$\}) = 0$.

Proof: Fix p $\in M$. $dim(T_p^* M) = k \Rightarrow$

$\exists \ \{g_1, ..., g_k\} \in \{f_1, ..., f_n\}$: span$(\{d_p g_1, ..., d_p g_k\}) = (T_p^* M) \Rightarrow$

a) $D_p(g_1, ..., g_k)$ is not degenerate. Inv. Map.Thm. $\Rightarrow$

b) $\exists V \subseteq M$, $L \subseteq R^k$: $\phi = (g_1, ..., g_k) : V \to L \subseteq R^k$ - diffeom.

Fix $(a_{k+1}, ..., a_n)$, $V \subseteq M$

$$(\{h_{k+1}, ..., h_n\} \cup \{g_1, ..., g_k\}) = \{f_1, ..., f_n\}$$

$$g_a := a_1 g_1 + ... + a_k g_k + a_{k+1} h_{k+1} + ... + a_n h_n,$$

$$g_a^* := g_a((\phi)^{-1}) = a_1 x_1 + ... + a_k x_k + a_{k+1} h_{k+1}^* + ... + a_n h_n^*.$$

Apply Lemma $\Rightarrow$

$$m(\{(a_1, ..., a_k) \in R^k : g_a^*|_L \text{ is not Morse}\}) = 0 \Rightarrow$$

$$m(\{(a_1, ..., a_k) \in R^k : g_a|_V \text{ is not Morse}\}) = 0.$$

Apply Fubbini $\Rightarrow$

$m(\{(a_1, ..., a_n) \in R^n : g_a|_V$ is not Morse$\}) = 0.$

Cover M by $\cup V_i \Rightarrow \forall\ V_i,\ \exists\ D_i \subseteq R^n : a \notin D_i \Rightarrow$

g_a is Morse. Take $\cup D_i$.

Corrolary

Given M - k-dim. manifold, $\exists f : M \to R$ Morse.

Reminder

$f : M \to R$ is Morse if for some chart $\phi : V \to R^n$, $f \circ (\phi)^{-1}$ is Morse.

Morse-Smale complex:

Sard ⇒ Smale

Given M-manifold, $\exists$ Riemannian metric, s.t.

$<,>$ is standard near crit. points; $\forall a, b$-crit. points, $(S(a) \pitchfork U(b))$

$$\frac{\partial x}{\partial t} = (gradf)(x) \rightarrow \begin{cases} S(a) := \{x : x(0) = x, lim_{t\rightarrow\infty} x(t) = a\} \\ U(b) := \{x : x(0) = x, lim_{t\rightarrow-\infty} x(t) = b\} \end{cases}$$

Want to construct differential complex:

$A_0 \to A_n \to ... \to A_1 \to A_0$, $A_i = Z^k$, $k = \#$ crit. p-ts of index i.

$d_i : A_i \to A_{i-1}$ - homomorphism, $d_i d_{i+1} = 0$.

Suppose dim(A_i)=n, dim(A_{i-1})=m.

$\{x_1, .., x_n\}$-basis of A_i, $\{y_1, .., y_m\}$-basis of A_{i-1}.

$d_i(x_s) = \Sigma_{j=1}^{m} a_{s_j} y_j$.

How to define coefficients? $d_{i-1}[a] = \Sigma n_b [b]$.