University of Toronto

MAT477 Seminar in Mathematics

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# Distance to Hypersurface Theorem

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# Theorem (Distance Formula)

For $P$ a polynomial and $a, x \in \mathbb{C}^n$,

$$\frac{1}{2n} d_P^*(a,x) \leq d_P(a,x) \leq \sqrt{n}\,\deg(P)\, d_P^*(a,x)$$

where

$$V(P,x) \coloneqq \{z \in \mathbb{C}^n | P(z) = P(x)\}$$

$$d_P(a,x) \coloneqq dist\big(a, V(P,x)\big)$$

$$d_P^*(a,x) \coloneqq \min_{1 \leq |\alpha| \leq \deg(P)} \left| \frac{P(a) - P(x)}{\frac{1}{\alpha!} D^\alpha P(a)} \right|^{\frac{1}{|\alpha|}}$$

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$$

$$|\alpha| \coloneqq \alpha_1 + \cdots \alpha_n$$

$$D^\alpha f \coloneqq \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} .$$

## Motivation for Theorem

Recall that polynomials of degree 1 defined on $\mathbb{C}^n$ satisfy

$$dist(a, \{z \in \mathbb{R}^n | P(z) = 0\}) = \frac{|P(a)|}{\|gradP(a)\|_{L^2}}$$

This is a special case of a similar well-known formula in $\mathbb{R}^n$.

$\|gradP(a)\|_{L^2} \approx \left|\max_j \frac{\partial P}{\partial x_j}(a)\right|$, with a constant that only depends on $n$.

Hence, when $\deg(P) = 1$

$$dist(a, \{z \in \mathbb{C}^n | P(z) = 0\}) \approx \frac{|P(a)|}{\left|\max\limits_j \frac{\partial P}{\partial x_j}(a)\right|}$$

## Preliminaries

Given any value of $P(x)$, we can consider the polynomial $Q(z) \coloneqq P(z) - P(x)$.

Hence, we may assume that $P(x) = 0$.

For $a, \xi \in \mathbb{C}^n$ with $|\xi| = 1$ we will define a new polynomial

$$P_\xi(z) \coloneqq P(z\xi + a).$$

If $P(a) = 0$ then $P(a) = P(x)$. Hence, $d_P(a, x) = 0$ and $d_P^*(a, x) = 0$ and so there is nothing to prove.

Thus, we assume that $P(a) \neq 0$ and so $P_\xi(0) \neq 0$.

Now, define the distance from $a$ to $V(P, x)$ along the complex line through $a$ in the direction $\xi$ to be:

$$dist_\xi\big(a, V(P, x)\big) \coloneqq \min_{z \in \mathbb{C}} \{|(z\xi + a) - a| : P(z\xi + a) = 0\}$$
$$= \min_{z \in \mathbb{C}} \big\{|z| : P_\xi(z) = 0\big\}.$$

# Outline of Proof

Claim 1:

$$dist_{\xi}\big(a, V(P,x)\big) \geq \frac{1}{2} \min_{1 \leq k \leq \deg(P)} \left| \frac{P_{\xi}(0)}{\frac{1}{k!} P_{\xi}^{(k)}(0)} \right|^{\frac{1}{k}}$$

Claim 2:

$$dist_{\xi}\big(a, V(P,x)\big) \leq \deg(P) \min_{1 \leq k \leq \deg(P)} \left| \frac{P_{\xi}(0)}{\frac{1}{k!} P_{\xi}^{(k)}(0)} \right|^{\frac{1}{k}}$$

Claim 3:

$$\sqrt{n}\, \deg(P)\, d_P^{*}(a,x) \geq d_P(a,x)$$

Claim 4:

$$\frac{1}{2n} d_P^{*}(a,x) \leq d_P(a,x)$$

**Proof of Claim 1:** $dist_{\xi}\big(a, V(P,x)\big) \geq \frac{1}{2}\min_{1 \leq k \leq \deg(P)} \left|\frac{P_{\xi}(0)}{\frac{1}{k!}P_{\xi}^{(k)}(0)}\right|^{\frac{1}{k}}$

First write $P_{\xi}(z)$ as a power series expansion $P_{\xi}(z) = \sum_{k=0}^{m} a_k z^k$ where $a_k = \frac{1}{k!}P_{\xi}^{(k)}(0)$. Then, $P_{\xi}(z) = 0$ implies

$$1 = -\sum_{k=1}^{m} \frac{a_k}{a_0} z^k$$

Hence,

$$1 \leq \sum_{k=1}^{m} \left|\frac{a_k}{a_0}\right| |z|^k \leq \sum_{k=1}^{m} (R|z|)^k$$

where

$$R \coloneqq \max_{1 \leq k \leq m} \left|\frac{a_k}{a_0}\right|^{\frac{1}{k}}.$$

Since $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$, we have

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1 \leq \sum_{k=1}^{m} (R|z|)^k \leq \sum_{k=1}^{\infty} (R|z|)^k$$

Hence, $R|z| \geq \frac{1}{2}$ and so

$$
\begin{aligned}
|z| &\geq \frac{1}{2}\frac{1}{R} \\
&\geq \frac{1}{2}\frac{1}{\max\limits_{1\leq k\leq m}\left|\frac{a_k}{a_0}\right|^{\frac{1}{k}}} \\
&\geq \frac{1}{2}\min_{1\leq k\leq m}\left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}
\end{aligned}
$$

Recall that

$$dist_\xi\big(a, V(P,x)\big) \coloneqq \min_{z\in\mathbb{C}}\big\{|z| : P_\xi(z) = 0\big\}.$$

Thus,

$$dist_\xi\big(a, V(P,x)\big) \geq \frac{1}{2}\min_{1\leq k\leq \deg(P)}\left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}$$

**Proof of Claim 2:** $dist_\xi\big(a, V(P,x)\big) \leq \deg(P) \min_{1\leq k\leq \deg(P)} \left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}$

Consider the polynomial $\tilde{P}_\xi(z) \coloneqq z^m P_\xi\left(\frac{1}{z}\right)$.

We can write $\tilde{P}_\xi(z)$ in two ways.

1) $\tilde{P}_\xi(z) = \sum_{j=0}^{m} a_{m-j} z^j$
2) $\tilde{P}_\xi(z) = a_0 \prod_{k=1}^{m}(z - z_k)$.

Equating coefficients yields,

$$\left|j!\, a_{m-j}\right| = \left|\frac{d^j}{dz^j}\tilde{P}_\xi(0)\right| \leq |a_0| \frac{m!}{(m-j)!} M^{m-j}$$

where

$$M \coloneqq \max_{1\leq k\leq m} |z_k| = \frac{1}{dist_\xi\big(a, V(P,x)\big)}.$$

Thus,

$$\left|\frac{\frac{1}{k!}P_\xi^{(k)}(0)}{P_\xi(0)}\right|^{\frac{1}{k}} \le mM, \qquad 1 \le k \le m,$$

and so it follows that

$$dist_\xi\big(a, V(P,x)\big) \le \deg(P) \min_{1\le k\le \deg(P)} \left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}.$$

Thus, we have proven

$$\frac{1}{2}\min_{1\le k\le \deg(P)} \left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}} \le dist_\xi\big(a, V(P,x)\big) \le \deg(P) \min_{1\le k\le \deg(P)} \left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}$$

which is equivalent to

$$\frac{1}{2} d_P^*(a,x) \le d_P(a,x) \le \deg(P)\, d_P^*(a,x)$$

To prove the general case we will make use of the fact that

$$dist\big(a,V(P,x)\big) = \min_{|\xi|=1} dist_\xi\big(a,V(P,x)\big).$$

We now make use of the Cauchy integral formula on polydiscs which states:

Let $D = \prod_1^n D_j$ be an open polydisc, let $\partial D = \prod_1^n \partial D_j$ denote its boundary, and let $f$ be continuous on the closure of $D$. Then,

$$f(z_1,\dots,z_n) = \frac{1}{(2\pi \mathrm{i})^{\mathrm{n}}}\int_{\partial D} \frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}\, d\zeta_1 \cdots d\zeta_n$$

By definition $P_\xi(z) = P(z\xi + a)$ and so $\frac{P_\xi^{(k)}(0)}{k!} = \sum_{|\alpha|=k} b_\alpha \xi^\alpha$where $b_\alpha = \frac{D^\alpha P(a)}{\alpha!}$.

$$\frac{1}{\alpha!}D^\alpha P(z\xi + a) = \frac{1}{(2\pi i)^n}\int\limits_{\xi\in T\times\cdots\times T} \frac{\frac{1}{k!}P_\xi^{(k)}(z)}{(\xi_1 - z_1)^{\alpha_1+1}\cdots(\xi_n - z_n)^{\alpha_n+1}} d\xi_1 \dots d\xi_n$$

By taking absolute values and evaluating at $z = 0$ we get,

$$\left|\frac{1}{\alpha!}D^{\alpha}P(a)\right| = \left|\frac{1}{(2\pi)^n}\int_{\xi\in T\times\cdots\times T}\frac{\frac{1}{k!}P_{\xi}^{(k)}(0)}{\xi^{\alpha}\xi_1\xi_2\cdots\xi_n}d\xi_1\ldots d\xi_n\right|$$

$$\leq \frac{1}{(2\pi)^n}\int_{\xi\in T\times\cdots\times T}\left|\frac{1}{k!}P_{\xi}^{(k)}(0)\right|d\xi_1\ldots d\xi_n$$

$$\leq \max_{\xi\in T\times\cdots\times T}\left|\frac{1}{k!}P_{\xi}^{(k)}(0)\right|$$

Thus, using the Cauchy integral formula, we have shown that for all $|\alpha| = k$

$$\left|\frac{1}{\alpha!}D^{\alpha}P(a)\right| \leq \max_{\xi\in T\times\cdots\times T}\left|\frac{1}{k!}P_{\xi}^{(k)}(0)\right|$$

**Proof of Claim 3:** $\sqrt{n}\ \deg(P)\, d_P^*(a,x) \geq d_P(a,x)$

We know that

$$\left|\frac{1}{\alpha!}D^\alpha P(a)\right| \leq \max_{\xi\in T\times\cdots\times T}\left|\frac{1}{k!}P_\xi^{(k)}(0)\right|$$

Then, setting $\xi' \coloneqq \frac{\xi}{\sqrt{n}}$ yields,

$$\begin{aligned}\left|\frac{1}{\alpha!}D^\alpha P(a)\right| &\leq \left(\sqrt{n}\right)^k \max_{|\xi|\leq 1}\left|\frac{1}{k!}P_\xi^{(k)}(0)\right| \\ &= \left(\sqrt{n}\right)^k \max_{|\xi|=1}\left|\frac{1}{k!}P_\xi^{(k)}(0)\right|.\end{aligned}$$

This implies,

$$\min_{|\alpha|=k}\left|\frac{P(a)}{\frac{1}{\alpha!}D^\alpha P(a)}\right|^{\frac{1}{|\alpha|}} \geq \frac{1}{\sqrt{n}}\min_{|\xi|=1}\left|\frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}.$$

Therefore,

$$d_P^*(a,x) \geq \frac{1}{\sqrt{n}} \min_{1\leq k\leq \deg(P)} \min_{|\xi|=1} \left| \frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)} \right|^{\frac{1}{k}}$$

$$= \frac{1}{\sqrt{n}} \min_{|\xi|=1} \min_{1\leq k\leq \deg(P)} \left| \frac{P_\xi(0)}{\frac{1}{k!}P_\xi^{(k)}(0)} \right|^{\frac{1}{k}}$$

$$\geq \frac{1}{\sqrt{n}} \min_{|\xi|=1} \frac{1}{\deg(P)} dist_\xi\big(a, V(P,x)\big)$$

$$= \frac{1}{\sqrt{n}} \frac{1}{\deg(P)} dist\big(a, V(P,x)\big)$$

Rearranging yields,

$$\sqrt{n}\ \deg(P)\ d_P^*(a,x) \geq d_P(a,x)$$

**Proof of Claim 4:** $\frac{1}{2n} d_P^*(a,x) \leq d_P(a,x)$

When $|\xi| = 1$ we have

$$\left|\frac{1}{k!} P_\xi^{(k)}(0)\right|^{\frac{1}{k}} = \left|\sum_{|\alpha|=k} \frac{D^\alpha P(a)}{\alpha!} \xi^\alpha\right|^{\frac{1}{k}} \leq \left(\sum_{|\alpha|=k} \left|\frac{D^\alpha P(a)}{\alpha!}\right|\right)^{\frac{1}{k}} \leq \max_{|\alpha|=k} \left|\frac{D^\alpha P(a)}{\alpha!}\right|^{\frac{1}{k}} \left(\sum_{|\alpha|=k} 1\right)^{\frac{1}{k}}$$

This implies,

$$\left|\frac{\frac{1}{k!} P_\xi^{(k)}(0)}{P_\xi(0)}\right|^{\frac{1}{k}} \leq n \max_{|\alpha|=k} \left|\frac{\frac{1}{\alpha!} D^\alpha P(a)}{P(a)}\right|^{\frac{1}{|\alpha|}}$$

After inverting, we get

$$\frac{1}{n} \min_{|\alpha|=k} \left|\frac{P(a)}{\frac{1}{\alpha!} D^\alpha P(a)}\right|^{\frac{1}{|\alpha|}} \leq \left|\frac{P_\xi(0)}{\frac{1}{k!} P_\xi^{(k)}(0)}\right|^{\frac{1}{k}}$$

Therefore,

$$\frac{1}{n} d_P^*(a,x) \leq \min_{1 \leq k \leq \deg(P)} \left|\frac{P_\xi(0)}{\frac{1}{k!} P_\xi^{(k)}(0)}\right|^{\frac{1}{k}} \leq 2 dist_\xi(a, V(P,x))$$

## Applications

The Distance Formula can be used when dealing with Markov inequalities to relate algebraic quantities purely in terms of geometric ones.

## Defintion

Suppose that $K \subset \mathbb{R}^n$ is closed. We say that K admits a local Markov inequality of exponent $\sigma \geq 1$ (in uniform norms) if there are constants $C_k, k = 0,1,2,\cdots$ such that for all polynomials P, $x \in K, 0 < \epsilon \leq 1$ and $0 \leq |\alpha| \leq \deg(P) \leq k$,

$$|D^\alpha P(x)| \leq C_0 (C_k \epsilon^{-\sigma})^{|\alpha|} \|P\|_{K \cap B_\epsilon(x)}$$

**Theorem**

The value $\sigma$ in the Markov inequality is directly related to the cuspidality of the set K.

Outline of Proof:

The Markov inequality can be restated as

$$|D^{\alpha}P(a)| \leq C_0(C_k\epsilon^{-\sigma})^{|\alpha|}\|P\|_{L_\infty(K\cap B_\epsilon(a))}$$

and also as

$$\max_{x\in K\cap B_\epsilon(a)} d_P^*(a,x) \geq \tilde{C}_k^{-1}\epsilon^{\sigma}$$

The distance formula allows us to replace the value $d_P^*(a,x)$ by $d_P(a,x)$ and so converting a purely algebraic quantitiy into an inequality involving the geometric concept of distance.

**Theorem**

There exist Sobolev inequalities and they are equivalent to Markov inequalities for an appropriate $\sigma$.

**Recommended Readings:**

P. Milman & L. Bos "A Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains ", Geometric And Functional Analysis, Volume 5, Number 6 p.853-923

D. Kinzebulatov "A note on Gagliardo-Nirenberg type inequalities on analytic sets", C. R. Math. Rep. Acad. Sci. Canada, Volume 30 (2009), p.97-105