Malgrange Preparation Theorem

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1. All Functions C^∞ All rings commutative with Id

Def: A germ is an equivalence class of functions:
representatives agree in the neighbourhood of some point.

E.g. $\mathcal{E}$(n) ring of C^∞ germs: $(\mathbb{R}^n, 0) \to \mathbb{R}$ at 0.

E.g. $\mathcal{E}$(n) $\supset$ m(n) $=$ $\{$f $\in \mathcal{E}(n)|$ f(0)=0 $\}$ is unique maximal ideal

E.g. f : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \Rightarrow f^* : \mathcal{E}(p) \to \mathcal{E}(n)$ homomorphism

Def: $\widetilde{f}$:$(\mathbb{R} \times \mathbb{R}^n$,0)$\ni$(t,x)$\to$f(t,x)$\in \mathbb{R}$, is p-regular with respect to t:

if $\widetilde{f}$(0,0) $= \frac{\partial}{\partial t}\widetilde{f}(0,0) = ... = \frac{\partial^{p-1}}{\partial t^{p-1}}\widetilde{f}(0,0) = 0$, but $\frac{\partial^p}{\partial t^p}\widetilde{f}(0,0) \neq 0$

2. Main Tools: Gen.Div.Thm and Nakayama Lemma

Generalized Division Theorem:Let $\widetilde{f}, \widetilde{g} \in \mathcal{E}(n+1)$ with $\widetilde{f}$ p-regular. Then $\exists \widetilde{Q} \in \mathcal{E}(n+1)$ and $\widetilde{h_j} \in \mathcal{E}(n)$, j=1,...,p, s.th. $\widetilde{g} = \widetilde{Q} \cdot \widetilde{f} + \widetilde{P_h}(t,x)$ where $\widetilde{P_h}(t,x) = \sum_{j=1}^{p} \widetilde{h_j}(x) t^{p-j}$

Nakayama Lemma Given ring $\mathcal{R}$ with unique maximal ideal m.

A is a fin.gen. $\mathcal{R}$-module, then m·A=A $\Rightarrow$ A=0.

Corollary: B is $\mathcal{R}$-module and B $\subset$ A then A = B+m·A $\Rightarrow$ A=B.

3. Malgrange-Mather Prep Thm:

I. **Thm.** For **A** finitely generated $\mathcal{E}(n)$ module:

A is finite over $\mathcal{E}(p) \Leftrightarrow$ the v.space $\dim_{\mathbb{R}}(\mathbf{A}/(f^*m(p) \cdot \mathbf{A})) < \infty$

II.**Corollary:** $\{a_1, ..., a_k\}$ generate **A** as an $\mathcal{E}(p)$-module

$\Leftrightarrow$ they generate the real vector space $\mathbf{A}/(f^*\mathrm{m(p)}\cdot\mathbf{A})$.

Remark :

- We'll see I $\Rightarrow$ II follows from Nakayama Lemma.
- We will often write $\mathcal{E}(p)$-module for $f^*\mathcal{E}(p)$-module.

4. Proof of Prep Thm I : $\Rightarrow$ is easy

$\mathbf{A}$ finite over $\mathcal{E}(p) \Rightarrow \bigoplus_{i=1}^{k} \mathcal{E}(p) \mapsto \mathbf{A}$ is epimorphism (onto)

$$\Rightarrow \exists\ \mathbb{R}^k \cong \bigoplus_{i=1}^{k} \mathcal{E}(p)/m(p) \mapsto \mathbf{A}/(f^* m(p) \cdot \mathbf{A})\text{is epi}$$

Note: $\mathbf{A} = < a_1, ..., a_k >_{\mathcal{E}(p)} \Rightarrow \mathbf{A}/(f^* m(p) \cdot \mathbf{A}) =$

$$\text{span}\{[a_1], ..., [a_k]\} \text{ over } \mathcal{E}(p)/m(p) = \mathbb{R}.$$

The other direction:

Case1: n=p+1 and $\widetilde{f} : (\mathbb{R} \times \mathbb{R}^p, 0) \ni (t, x) \to (x) \in (\mathbb{R}^p, 0)$

is the projection onto the second factor.

5. Proof of the Prep Thm (Case1 : n=p+1)

Say $\mathbf{A} = < a_1, ..., a_k >_{\mathcal{E}(p+1)}$, $\mathbf{A}/(f^* m(p) \cdot \mathbf{A}) = < a_1, ..., a_k >_{\mathbb{R}}$.

$\Longrightarrow \forall \mathbf{a} \in \mathbf{A}$, $\mathbf{a}{=}\sum_{j=1}^{k} c_j a_j + ba'$, where $c_j \in \mathbb{R}$, $b \in f^* m(p)$

and $\mathbf{A} \ni a' = \sum_{j=1}^{k} g_j a_j$, for $g_j \in \mathcal{E}(p+1)$;

Say $z_j{:=}bg_j \in f^* m(p) \cdot \mathcal{E}(p+1) {\Rightarrow} \mathbf{a}{=}\sum_{j=1}^{k} c_j a_j + \sum_{j=1}^{k} z_j a_j$

For $\mathbf{a}{=}ta_i \Rightarrow$ $ta_i{=}\sum_{j=1}^{k}(c_{ij} + z_{ij})a_j$ $\qquad (\sharp)$

Say (δ_{ij}):=identity matrix and $\vec{a} := (a_1, ..., a_k)$. Then $(\sharp) \Longrightarrow$

6. Still case 1

$$(t\delta_{ij} - c_{ij} - z_{ij})\cdot\vec{a} = 0. \qquad (\sharp\sharp)$$

Say (b_{ij}):=$(\mathrm{t}\delta_{ij} - c_{ij} - z_{ij})$ and $(B_{ij}) := (b_{ij})^{adjoint} \Longrightarrow$

$$(B_{ij}) \cdot (b_{ij}) = \det(b_{ij}) \cdot (\delta_{ij}) \qquad (*)$$

$\det(\mathrm{t}\delta_{ij} - c_{ij} - z_{ij}) =: \triangle(t, x)$ is a function of (t,x).

$(\sharp\sharp)$ and $(*) \Rightarrow \triangle \cdot \vec{a} = 0$

Since $z_{ij}(t, 0) = 0$ and $\triangle$ is a monic polynomial in t $\Longrightarrow$

$\det(t\delta_{ij} - c_{ij} - z_{ij})$is **q-regular** *at (t,0) for $q \leq k$.*

7.End of step1

$$\triangle \cdot \vec{a}=0 \Rightarrow \triangle \cdot \mathbf{A} = 0 \Rightarrow$$

A is a finite $\mathcal{E}(p+1)/\triangle \cdot \mathcal{E}(p+1)$-module (assumed, n=p+1).

since $\triangle$ *is q-regular, the Generalized Division Thm* $\Longrightarrow$

$$\mathcal{E}(p+1)/\triangle \cdot \mathcal{E}(p+1) \;=< 1, t, ..., t^{q-1} >_{\mathcal{E}(p)}.$$

since **A** *finite over* $\mathcal{E}(p+1)/\triangle\,\mathbf{.}\,\mathcal{E}(p+1)$ *and*

$\mathcal{E}(p+1)/\triangle\,\mathbf{.}\,\mathcal{E}(p+1)$ *is finite over* $\mathcal{E}(p)$

$\Rightarrow$ **A** *is finite over* $\mathcal{E}(p)$, completing **case 1**.

8. Case: $\tilde{f}$ has rank=n≤p and on ...

Rank Thm: $\exists$ coordinates s.th. $\tilde{f}$:$(x_1,...,x_n)\rightarrow(x_1,...,x_n,0,...,0)$

$\mathbb{R}^n \hookrightarrow \mathbb{R}^p \Rightarrow$ germs $\tilde{\phi}:(\mathbb{R}^n,0)\rightarrow\mathbb{R}$ extend to $(\mathbb{R}^p,0)$, i.e

$f^*:\mathcal{E}(p)\mapsto\mathcal{E}(n)$ is surjective and so
$\mathbf{A}=<a_1,...,a_k>_{\mathcal{E}(n)} \Longrightarrow \mathbf{A}=<a_1,...,a_k>_{\mathcal{E}(p)}$.

General Case: maps $\tilde{f}:(\mathbb{R}^n,0)\rightarrow(\mathbb{R}^p,0)$ are composites:

$$(\mathbb{R}^n,0)\xrightarrow{(id,\tilde{f})}(\mathbb{R}^n\times\mathbb{R}^p,0)\xrightarrow{pr_2}(\mathbb{R}^p,0)$$

(immersion followed by a sequence of n projections of **case1**).

9. Hence suffices to show (Prep Thm conclusion):

Property $\mathbb{M}(\tilde{f})$ $\quad dim_{\mathbb{R}}\mathbf{A}/(f^*m(p)\cdot\mathbf{A}) < \infty \Rightarrow \mathbf{A}$ finite over $\mathcal{E}(p)$

$$(\mathbb{R}^n, 0) \xrightarrow{\tilde{f}} (\mathbb{R}^p, 0) \xrightarrow{\tilde{g}} (\mathbb{R}^q, 0)$$

$$\mathbb{M}(\tilde{f}) \text{ and } \mathbb{M}(\tilde{g}) \Rightarrow \mathbb{M}(\tilde{g}\circ\tilde{f}).$$

Assume $\mathbf{A}$ finite over $\mathcal{E}(n)$ and $dim_{\mathbb{R}}(\mathbf{A}/(\tilde{g}\circ\tilde{f})^*m(q)\cdot\mathbf{A}) < \infty$.

$$f^*g^*m(q) \subset f^*m(p) \Rightarrow \dim(\mathbf{A}/f^*m(p)\cdot\mathbf{A}) < \infty$$

Therefore using $\mathbb{M}(\tilde{f}) \Rightarrow \mathbf{A}$ is finite over $\mathcal{E}(p)$.

Next: $\mathbf{A}/g^*m(q)\cdot\mathbf{A}$=$\mathbf{A}/f^*g^*m(q)\cdot\mathbf{A}$, and is of finite dim.

Hence, $\mathbb{M}(\tilde{g}) \Rightarrow \mathbf{A}$ is finite over $\mathcal{E}(q)$. Done. $\surd$

10. Prep Thm II: $< a_1, ..., a_k >_{\mathbb{R}} = \mathbf{A} \Leftrightarrow < a_1, ..., a_k >_{\mathbb{R}} = \mathbf{A}/(f^* \mathrm{m(p)} \cdot \mathbf{A})$

Proof: $\Rightarrow$ is easy(as on page 4). Next $\Leftarrow$:

By Malg-Math Prep $\mathbf{A}/(f^* m(p) \cdot \mathbf{A}) = < a_1, ..., a_k >_{\mathbb{R}}$

$\Longrightarrow \mathbf{A}$ is finite over $\mathcal{E}(p)$.

In particular, $\mathbf{A} = < a_1, ..., a_k >_{\mathcal{E}(p)} + \mathrm{m(p)} \cdot \mathbf{A}$.

Nakayama Lemma $\Rightarrow \mathbf{A} = < a_1, ..., a_k >_{\mathcal{E}(p)}$. $\surd$

11. Special Case : $\mathbf{A} = \mathcal{E}(n)$

Def: $\mathcal{E}(\mathrm{n}) \supset m(n)^k := k^{th}$ power of m(n); $m(n)^\infty := \bigcap_{j=1}^{\infty} m(n)^j$

Def: $\widehat{\mathcal{E}(n)} := \mathcal{E}(\mathrm{n})/m(n)^\infty = \mathbb{R}[[x_1, x_2, ..., x_n]] \ni \hat{f} \leftarrow \tilde{f} \in \mathcal{E}(n)$

$\Rightarrow \widehat{\mathcal{E}(n)} \supset \widehat{m(n)} := \{\hat{f} \in \widehat{\mathcal{E}(n)} | f(\hat{0}) = 0\}$ the unique max ideal.

Then $\mathcal{E}(\mathrm{n}) \ni \widetilde{f} \rightarrow \hat{f} \in \widehat{\mathcal{E}(n)}$ is the Taylor homomorphism and:

(i) $\widehat{f \cdot g} = \hat{f} \cdot \hat{g}$;

(ii) let $\hat{f}$=p+m and $\hat{g}$=q+r, p, q Taylor polynomials of deg k

and m,r$\in m(n)^{k+1} \Rightarrow \hat{f} \cdot \hat{g} = \widehat{f \cdot g} = \widehat{p \cdot q} = \mathrm{p} \cdot q$ (mod $m(n)^{k+1}$).

12. Preparation Theorem $\mathbf{A} = \mathcal{E}(n)$

Preparation Theorem (in the Malgrange Form):

f^* induces the homomorphism of power series $\hat{f}^*:\widehat{\mathcal{E}(p)} \to \widehat{\mathcal{E}(n)}$

The following statements are equivalent:

1. $\phi_1, \phi_2, ..., \phi_k \in \mathcal{E}(n)$ generate $\mathcal{E}(n)$ as an $f^*\mathcal{E}(p)$ module.
2. $\hat{\phi}_1, \hat{\phi}_2, ..., \hat{\phi}_k$ generate $\widehat{\mathcal{E}(n)}$ as an $\hat{f}^*\widehat{\mathcal{E}(p)}$-module.
3. $\phi_1, \phi_2, ..., \phi_k$ generate $\mathcal{E}(n)/f^*m(p) \cdot \mathcal{E}(n)$ over $\mathbb{R}$.
4. $\hat{\phi}_1, ..., \hat{\phi}_k$ are the generators of $\widehat{\mathcal{E}(n)}/\hat{f}^*\widehat{m(p)} \cdot \widehat{\mathcal{E}(n)}$ over $\mathbb{R}$.

13. Easy implications

- 1⇒2 direct, 1⇔3 is the Malg. Preparation theorem.

- 2⇒4 is easy and direct from def.

- 3⇒4 Since $\mathcal{E}(n) = \phi_1 \cdot \mathbb{R} + ... + \phi_k \cdot \mathbb{R} + f^* m(p) \cdot \mathcal{E}(n)(\diamond)$

apply Taylor homomorphism $\mathcal{T} : \mathcal{E}(n) \to \widehat{\mathcal{E}(n)}$, which is onto (Thm. due to Borel) $\Longrightarrow \hat{\phi}_1 \cdot \mathbb{R}+...+\hat{\phi}_k \cdot \mathbb{R}+\hat{f}^* \widehat{m(p)} \cdot \widehat{\mathcal{E}(n)}=\widehat{\mathcal{E}(n)}$.

- 4⇒3 $dim_{\mathbb{R}}\mathcal{E}(n)/(...) < \infty$, where $(...){:=}(m(p) \cdot \mathcal{E}(n) + m(n)^{\infty})$.

14. Proof of 4 $\Rightarrow$ 3, i.e (formal) $\Rightarrow$ (in germs)

$$\Longrightarrow \exists \text{ k such that } m(n)^k + (...)/(...) = m(n)^{k+1} + (...)/(...)$$

$$\Longleftrightarrow m(n)^k + (...) = m(n)^{k+1} + (...) \ (**)$$

$\Rightarrow m(n)^k$+B $= m(n)^{k+1}$+B,where B:=m(p)$\mathcal{E}(n) \subset$

A:=$m(n)^k$+B $\Longrightarrow$ A = B + m(n)A and

Nakayama Lemma $\Rightarrow$ B = A, i.e. m(p)$\mathcal{E}(n)$= $m(n)^k$+m(p)$\mathcal{E}(n)\Rightarrow$

m(p)$\mathcal{E}(n)$=(...) $\Rightarrow \mathcal{E}(n)$/m(p)$\mathcal{E}(n) \cong \mathcal{E}(n)/(...) \cong \widehat{\mathcal{E}(n)}/\widehat{m(p)}\widehat{\mathcal{E}(n)}$

and is, therefore, generated by $\phi_1, \phi_2, ..., \phi_k$ over $\mathbb{R}$. $\square$

15.Proof of Nakayama's Lemma and its corollary

Cor: $A \subset B + m \cdot A \Rightarrow A/B \subset (B+mA)/B = m(A/B)$.

By Nakayama, $A/B = 0$ and so $A = B$. $\surd$

Proof of the Lemma: $\forall z \in m$, $1+z$ is a unit.

If $A = < a_1, ..., a_n >_{\mathcal{R}}$, then $\exists z_{ij} \in m$ s.th. $a_i = \sum_{j=1}^{n} z_{ij} a_j$.

In matrix form, a=Za, i.e (Z-Id)a=0.

det(Z-Id) is the char. polynomial at 1 and $= \pm 1 + z$, $z \in m$

$\Longrightarrow$ Z-Id is invertible. Hence $a = (a_1, ..., a_n) = 0$ and $A = 0$. $\surd$