C-infinity Preparation Theorem

OMAR RAMMO

University of Toronto
Department of Mathematics

March 5, 2010

1. C^∞ Preparation and Division Theorems

All functions below are C^∞ near 0 with $x \epsilon \mathbb{R}^n$ and $t \epsilon \mathbb{R}$

Prep Thm: $f(x,t)_{x=0} = t^d h(t)$, $h(0) \neq 0 \Longrightarrow \exists q_f$ and λ_i s.th.

$\lambda_i(0) = 0$, $q_f(0) \neq 0$, $f(x,t) = P^d(t,\lambda) \cdot (q_f(x,t))$

where $P^d(t,\lambda) = t^d + \sum_{i=1}^{d} \lambda_i(x) t^{d-i}$

Div Thm: $\forall f(x,t)$, d$\epsilon\mathbb{N}$, $\lambda \epsilon \mathbb{R}^d \Longrightarrow \exists Q_f$ and $r_{j,f}$ s.th.

$f(x,t) = P^d(t,\lambda) \cdot Q_f(x,t,\lambda) + \sum_{j=1}^{d} r_{j,f}(x,\lambda) t^{d-j}$ $(\star)$

Div Thm $\Rightarrow$Prep Thm: solve $r_{i,f}(x, f(x)) = 0$ and $\lambda_i(0) = 0$ $\forall i$

2. Proof (Via Implicit Function Theorem)

Show 1. (i) $r_{i,f}(0,0) = 0$ (ii) $Q_f(0,0,0) \neq 0$ and

2. $D = det(\frac{\partial r_{i,f}}{\partial \lambda_j})_{1 \leq i,j \leq d} \neq 0$

1. (i) Set $x = 0$, $\lambda = 0$ and compare orders of vanishing in t at 0.

$f(0,t) = t^d h(t) \Rightarrow r_{j,f}(0,0) = 0 \ \forall j$ and $Q_f(0,t,0) = h(t)$

$\Rightarrow Q_f(0,0,0) = h(0) \neq 0$

(ii) Apply $\left[\frac{\partial}{\partial \lambda_j}\right]_{x=0,\lambda=0}$ to $(\star)$ $\Rightarrow$ upper triangular matrix with

diagonal entries$=$ $Q_f(0,0,0) \Rightarrow D \neq 0$, Done.

3. Reduction of Division Theorem to Thm 1

$V^d := \{(t,\lambda) : P^d(t,\lambda) = 0\}$; $\pi_d : V^d \ni (t,\lambda) \longmapsto \lambda \epsilon \mathbb{R}^d$

Def: Let $C^\infty_\pi(V^d \times \mathbb{R}^n)$ be the subspace of $C^\infty(V^d \times \mathbb{R}^n)$

consisting of all functions constant on the fibers $\pi_d^{-1}(\lambda)$.

Theorem 1: $\exists\ J : C^\infty_\pi(V^d \times \mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{d+n})$ s.th.

$\forall\ (t,\lambda)\epsilon V^d$, $x \epsilon \mathbb{R}^n$: $(J\phi)(\lambda, x) = \phi(t,\lambda,x)$

Define λ^d by $P^d(z, \lambda^d(s,\mu)) = (z - s)\cdot P^{d-1}(z,\mu)$

$\Longrightarrow \lambda_1 = -s + \mu_1$, $\lambda_j = -\mu_{j-1}\cdot s + \mu_j$, and $\lambda_d = -\mu_{d-1}\cdot s$

Note: $(s,\mu) \longmapsto (s, \tilde{\lambda}(s,\mu))$ is invertible change of coordinates

4. Application: General C^∞ Division of g(x,t) by f(x,t)

Gen. C^∞ Div. Thm If f is s.th. $f(0,t) = t^d h(t)$, $h(0) \neq 0$

$\Longrightarrow \exists$ G and R_j such that $g = G \cdot f + \sum_{j=1}^{d} R_j(x) t^{d-j}$

Proof: Prep Thm for f and Div. Thm for g with $d = d_f$

$\Longrightarrow (t^d + \sum_{j=1}^{d} \lambda_i(x) t^{d-i}) = \frac{f(x,t)}{q_f(x,t)}$

Apply Div. Thm to g and plug in $\lambda_j = \lambda_j(x)$ from above.

$g = f \cdot \frac{Q_g(x,t,f(x))}{q_f(x,t)} + \sum_{j=1}^{d} r_{j,g}(x, f(x)) t^{d-j}$ Done.

5. Proof of Thm 1 implies Div. Thm with assertion:

$$Q_f^d(t, \lambda^d(s,\mu)) = \frac{1}{t-s}\left[Q_f^{d-1}(t,\mu) - Q_f^{d-1}(s,\mu)\right] \ (\star\star)$$

Proof by induction on $d \geq 1$:

<u>Case $d = 1$</u>: $f(x,t) = (t-\lambda) \cdot \frac{f(x,t)-f(x,\lambda)}{t-\lambda} + f(x,\lambda)$

$\frac{g(t)-g(0)}{t} = \int_0^1 g'(st)ds$; $(\nabla f)(t) := \frac{f(t)-f(s)}{t-s}$

$P(t,\lambda) := P^d(t,\lambda)$, $P(t,\mu) := P^{d-1}(t,\mu)$, $P(t,\nu) := P^{d-2}(t,\nu)$

Indexes of $\lambda^d(s,\mu)$, $\mu^{d-1}(\tau,\nu)$ we skip; $\tilde{\lambda} := (\lambda_1, ..., \lambda_{d-1})$

6. Proof of the inductive 'step':

i.e. : True for $2, ..., d-1 \Longrightarrow$ for d ($\lambda \epsilon \mathbb{R}^d$, $\mu \epsilon \mathbb{R}^{d-1}$, $\nu \epsilon \mathbb{R}^{d-2}$).

$P(z, \lambda(s, \mu(\tau, \nu))) = (z-s)(z-\tau) \cdot P(z, \nu)$

$\Rightarrow \lambda(s, \mu(\tau, \nu))$ is symmetric in (s, τ)

Now, true for d-1 $\Rightarrow$ formula ('almost' as required)

$f(t) = \frac{Q_f^{d-1}(t,\mu) - Q_f^{d-1}(s,\mu)}{t-s} \cdot P(t, \lambda(s, \mu)) + \sum_{k=1}^{d} r_{k,f}(s, \mu) t^{d-k}$ (1)

$(\nabla Q_f^{d-1})(t, s, \mu(\tau, \nu)) := \frac{\frac{Q_f^{d-2}(t,\nu) - Q_f^{d-2}(\tau,\nu)}{t-\tau} - \frac{Q_f^{d-2}(s,\nu) - Q_f^{d-2}(\tau,\nu)}{s-\tau}}{t-s}$

$\Longrightarrow (\nabla Q_f^{d-1})(t, s, \mu(\tau, \nu)) = (\nabla Q_f^{d-1})(t, \tau, \mu(s, \nu))$

7. Proof of the inductive 'step' (continued)

Therefore by (1) and the symmetry of $\lambda(s, \mu(\tau, \nu))$ in (s, τ),

$\Longrightarrow r_{k,f}(s, \mu(\tau, \nu)) = r_{k,f}(\tau, \mu(s, \nu))$ *for* $1 \leq k \leq d$ $(*)$

Recall $(s, \mu) \longmapsto (s, \tilde{\lambda}(s, \mu))$ is invertible change of coordinates.

Let $(s, \tilde{\lambda}) \longmapsto (s, \eta(s, \tilde{\lambda}))$ where $\mu = \eta(s, \tilde{\lambda})$ be the inverse.

$(s, \tilde{\lambda})$ are global polynomial coordinates on V^d, so functions

$\tilde{r}_{k,f}(s, \tilde{\lambda}) := r_{k,f}(s, \eta(s, \tilde{\lambda}))$ are on V^d (and are in $C^\infty_\pi(V^d)$!)

Now, suppose $s \neq \tau$, $(s, \tilde{\lambda})$ and $(\tau, \tilde{\lambda}) \in \pi_d^{-1}(\lambda) \cap V^d$

8. Thm 1 $\Longrightarrow$ Div. Thm (remainders $\in C_\pi^\infty(V^d)$!)

$\exists \nu \epsilon \mathbb{R}^{d-2}$ s.th. $P^d(t,\lambda) = (t-s)(t-\tau)\cdot P^{d-2}(t,\nu)$,

where $\lambda = \lambda(s,\mu(\tau,\nu))$. Symmetry of λ and $(*)$ $\Longrightarrow$

$\eta(s,\tilde{\lambda}) = \mu(\tau,\nu)$, $\eta(\tau,\tilde{\lambda}) = \mu(s,\nu) \Longrightarrow$ remainders $\in C_\pi^\infty(V^d)$:

$\tilde{r}_{k,f}(s,\tilde{\lambda}) = r_{k,f}(s,\mu(\tau,\nu)) = r_{k,f}(\tau,\mu(s,\nu)) = \tilde{r}_{k,f}(\tau,\tilde{\lambda})$.

Now, Thm1 $\Longrightarrow r^d_{k,f}(\lambda) = (J\tilde{r}_{k,f})(\lambda)$, where

$r^d_{k,f}(\lambda(s,\mu)) = (J\tilde{r}_{k,f})(\lambda(s,\mu)^d_{k,f}) = \tilde{r}_{k,f}(s,\tilde{\lambda}(s,\mu)) = r_{k,f}(s,\mu)$

$P^d(t,\lambda) = 0 \Rightarrow \lambda = \lambda(t,\mu)$ for some μ, and (1) $\Longrightarrow$

$f(t) - \sum_{k=1}^d r^d_{k,f}(\lambda)\cdot t^{d-k} = f(t) - \sum_{k=1}^d r_{k,f}(t,\mu)\cdot t^{d-k} = 0(\diamondsuit)$

9. Completion of proof Thm 1 implies Div. Thm.

Let $f(t,\lambda) := f(t) - \sum_{k=1}^{d} r_{k,f}^{d}(t,\lambda) \cdot t^{d-k}$

Applying change of coordinates: $(t,\lambda) \longmapsto (t, \tilde{\lambda}, P^d(t,\lambda))$ gives

in new coordinates $\phi(t,\tilde{\lambda},p) := f(t;\tilde{\lambda}; p - t^d - \sum_{k=1}^{d} \lambda_k \cdot t^{d-k})$

$(\diamondsuit) \Rightarrow \phi(t,\tilde{\lambda},0) = 0$. Therefore, $\phi(t,\tilde{\lambda},p)$ is divisible by p, and

$f(t,\lambda)$ is divisible by $P^d(t,\lambda) \Longrightarrow (\star)$.

Now, (1) with $P^d(t,\lambda(s,\mu)) \neq 0 \Longrightarrow (\star\star)$ (i.e. extra assertion) $\square$

Abusing notation, I'll skip indicating depencence on parameter x:

10. Sketch of Proof of Thm 1, i.e. $J\phi \in C^\infty(\mathbb{R}^d)_{\restriction_{V^d}}$

$\pi_d : V^d \to \mathbb{R}^d$ is proper and local diffeomorphism on set

$U := \{(t, \lambda) \in V^d : \frac{\partial P(t,\lambda)}{\partial t} \neq 0\} \implies J\phi \in C^\infty(\tilde{U})$, $\tilde{U} := \pi_d(U)$.

Plan: Show all derivatives $D^\alpha J\phi$ extend to $\pi_d(V^d)$ as C^0 via

proving by induction on $|\alpha|$ that $(D^\alpha J\phi) \circ \pi_d \in C^\infty(V^d)$.

Suffices to show: **1**. $J\phi \epsilon C^1(\pi_d(V^d))$ **2**. $(d_\lambda J\phi) \circ \pi_d \epsilon C^\infty(V^d)$.

Note: $\pi_{2k+1}(V^{2k+1}) = \mathbb{R}^{2k+1}$, and $\mathbb{R}^{2k} \backslash \pi_{2k}(V^{2k})$ is convex, $k \epsilon \mathbb{N}$

$\implies$ would follow by Whitney C^∞-Extention Thm. that

$J\phi$ extends to $\mathbb{R}^d$ as a C^∞ function, as required.

11. Proofs of 1. and 2.

$$\psi(t,\tilde{\lambda}) := J\phi(\pi_d(t,\tilde{\lambda})) \Longrightarrow (d_\lambda J\phi)(\lambda) \cdot \frac{\partial(\lambda_1,\ldots,\lambda_d)}{\partial(t,\lambda_1,\ldots,\lambda_{d-1})} = d_{(t,\tilde{\lambda})}\psi$$

$$\Longleftrightarrow (d_\lambda J\phi)(\pi_d(t,\lambda)) \cdot$$

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -\frac{\partial P(t,\lambda)}{\partial t} & -t^{d-1} & -t^{d-2} & \ldots & -t^2 & -t \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi(t,\tilde{\lambda})}{\partial t} \\ \frac{\partial\psi(t,\tilde{\lambda})}{\partial\lambda_1} \\ \ldots \\ \ldots \\ \ldots \\ \frac{\partial\psi(t,\tilde{\lambda})}{\partial\lambda_{d-1}} \end{pmatrix}$$

2 things to show: (i) The function $\phi^{new}(t,\lambda) := (d_\lambda J\phi)(\pi_d(t,\lambda))$

as a function on V^d coincides with a d-tuple of C^∞ functions,

for which solving the system above suffices to show that

12. $\frac{\partial\psi}{\partial t}(t,\tilde{\lambda})/\frac{\partial P}{\partial t}(t,\tilde{\lambda})$ is C^∞ . With $(t,\lambda)\in V^d$:

$\frac{\partial P(t,\lambda)}{\partial t} = 0 \Rightarrow \exists$ sequences $\{V^d \ni (s_{jn},\lambda_n) \longrightarrow (t,\lambda)\}_{j=1,2}$

with $s_{1n} \neq s_{2n} \Longrightarrow \frac{\partial\psi(t,\tilde{\lambda})}{\partial t} = lim_{n\to\infty}\frac{\psi(s_{1n},\tilde{\lambda}_{(n)})-\psi(s_{2n},\tilde{\lambda}_{(n)})}{s_{1n}-s_{2n}} = 0$.

Now coord change $(t,\lambda_1,...,\lambda_{d-1}) \mapsto (t,\lambda_1,...,\lambda_{d-2},p_1 = \frac{\partial P}{\partial t})$

and let $\theta(t,\lambda_1,...,\lambda_{d-2},p_1) := \frac{\partial}{\partial t}\psi(t,\lambda_1,...,\lambda_{d-1})$. Now,

$\theta(t,\lambda_1,...,\lambda_{d-2},0) = 0 \Rightarrow \theta(t,\lambda_1,...,\lambda_{d-2},p_1)$ is divisible by p_1,

$\Rightarrow \frac{\partial\psi}{\partial t}(t,\tilde{\lambda})/\frac{\partial P}{\partial t}(t,\tilde{\lambda})$ is C^∞, which completes the proof of (i)

13. Pair $(J\phi, J\phi^{new})$ is a Whitney C^1-function on $\mathbb{R}^d$

$(d_\lambda J\phi) \circ \pi_d$ is C^∞ on V^d and with π_d proper $\Longrightarrow$

$d_\lambda J\phi$ extends as C^0 from $\tilde{U}$ to $\pi_d(V^d)$, and is $J\phi^{new}$.

Let $\gamma := \{\lambda \epsilon \pi_d(V^d) :\ s.th.\ \exists a \epsilon \mathbb{R},\ P^d(z,\lambda) = (z-a)^d\}$.

Claim: $(J\phi)\epsilon C^1(\pi_d(V^d)\backslash\gamma)$.

Proof: Induction on d using 'resultants' (details in Baxter's talk)

Consider $(t^!, \lambda^!)$ s.th. $P^d(z,\lambda^!) = (z-t^!)^l \cdot P^{d-l}(z,\eta^!)$,

where $l < d$, $t^! \in \mathbb{R}$, $\lambda^! \epsilon \mathbb{R}^d$, $\eta^! \epsilon \mathbb{R}^{d-l}$, and $P^{d-l}(t^!, \eta^!) \neq 0$

14. Proof of Claim (induction on $l < d$)

$\Longrightarrow P^d(z, \lambda(\xi, \eta)) = P^l(z, \xi) \cdot P^{d-l}(z, \eta)$ defines the map

$(\xi, \eta) \longmapsto \lambda(\xi, \eta)$, a loc. diffeo. near $(\xi^!, \eta^!)$ s.th. $P^{d-l}(t^!, \eta^!) \neq 0$

and $P^l(z, \xi^!) := (z - t^!)^l$ (due to resultants theory...).

With this change of variables $(t, \xi, \eta) \longmapsto (t, \lambda(\xi, \eta))$ in

neighbourhoods of $(t^!, \xi^!, \eta^!) \in V^l \times \mathbb{R}^{d-l}$ and of $(t^!, \lambda^!) \in V^d$

$\Longrightarrow P^{d-l}(t^!, \eta^!) \neq 0 \Longrightarrow P^{d-l}(t, \eta) \neq 0$ near $(t^!, \eta^!)$ and,

$0 = P^d(t, \lambda(\xi, \eta)) = P^l(t, \xi) \cdot P^{d-l}(t, \eta) \Longrightarrow \; P^l(t, \xi) = 0$.

15. End of proof of Claim (summed up in a diagram)

$$\begin{array}{ccc} V^d_{(t,\lambda)} & \longrightarrow & V^l_{(t,\xi)} \times \mathbb{R}^{d-l}_\eta \\ \downarrow & & \downarrow \\ \mathbb{R}^d_\lambda & \longrightarrow & \mathbb{R}^l_\xi \times \mathbb{R}^{d-l}_\eta \end{array}$$

By the inductive hypothesis $(l < d)$ claim follows.

It remains to show $J\phi$ is C^1 on $\pi_d(V^d)$ including curve γ ,

i.e. when $P^d(z, \lambda) = (z - a)^d$ for $a \in \mathbb{R}$.

$P^d(z, \lambda_a) := P^d((z - a), \lambda)$ defines a diffeomorphism $\lambda_a \mapsto \lambda$

$\Longrightarrow$ Enough to prove differentiability at $0 \in \mathbb{R}^d$

16. Proof of Theorem 1 (conclusion)

Given $\lambda \in \pi_d(V^d)$, $\exists$ smooth path connecting λ and 0 s.th.

$\lambda(s) \in \pi_d(V^d) \backslash \gamma \ \forall s \in (0,1)$ with length $\leq$ const. $\cdot \sqrt{\sum_{k=1}^{d} \lambda_k^2}$.

$\lim_{s \to 1} (J\phi)(\lambda(s)) - (J\phi)(\lambda(1-s)) = \int_{1-s}^{s} (d_\lambda J\phi)(\lambda(\tau)) \cdot \frac{d\lambda}{d\tau}(\tau) d\tau$

$\Longrightarrow \ (J\phi)(\lambda) - (J\phi)(0) = \int_0^1 (d_\lambda J\phi)(\lambda(\tau)) \cdot \frac{d\lambda}{d\tau}(\tau) d\tau$

With estimate on the length of the path differentiability of

$(J\phi)(\lambda)$ at $\lambda = 0$ follows, which completes the proofs.