Hirzebruch-Riemann-Roch Theorem in Dimension One via Mumford

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The HRR Theorem

In all slides, X denotes a smooth complex projective curve in $\mathbb{P}^r$.

Definition : Let P_X be the Hilbert polynomial associated with X .

The arithmetic genus of X is $g_a(X) := (-1)(P_X(0) - 1)$.

HRR Theorem : If g is the top. genus of X , then $g_a(X) = g$.

Notation : Let $e(X)$ be the Euler characteristic of X and $d :=$ $\deg X$. For all $x \in X$, $E^*_{x,X}$ is the tangent cone at x . For a linear space L and $x \in X \cap L$, $i(x; X \cap L)$ is the intersection multiplicity. For a linear space L or point x , let p_L , resp. p_x , denote the projection with center L , resp. $\{x\}$.

Idea of Proof

Step 1 : Consider projections $p_k := p_L : X \to \mathbb{P}^k$ where L is of dimension $r - k - 1$ and disjoint from X .

Claim : For a.e. L ,

a) if $k \geq 3$, then $p_k(X)$ is birational to X and is smooth;

b) if $k = 2$, then $p_2(X)$ is birational to X and smooth except for finitely many ordinary double points;

c) if $k = 1$, then $p_1 : X \to \mathbb{P}^1$ is a covering of degree d and smooth except for finitely many ordinary branch points.

Step 2 : If $p_2(X)$ has δ double points and p_1 has β branch points, then:

a) $g_a(X) = \frac{1}{2}(d-1)(d-2) - \delta$;

b) $d(d-1) = \beta + 2\delta$;

c) $e(X) = 2d - \beta$.

As smooth curves are compact oriented 2-manifolds, we have $e(X) = 2 - 2g$. Then using these four formulas, we find that $g_a(X) = g$.

Step 1a: Projecting X to $\mathbb{P}^3$

We first show if $r > 3$, then for a.e. $x \in \mathbb{P}^r \setminus X$, $p_x(X)$ is birational to X and smooth. It suffices to find an x s.th. $\forall u \in X$, $\overline{ux} \cap X = \{u\}$ and $\overline{ux} \neq E^*_{u,X}$. If so, then $p_x : X \to p_x(X)$ is bijective, hence birational, and by Corollaries 5.14 and 5.15, $p_x(X)$ would be smooth. To find such an x , it suffices for $x \notin S :=$ $[\bigcup_{u \neq v \ ; u,v \in X} \overline{uv}] \cup [\bigcup_{u \in X} E^*_{u,X}]$. It now suffices to show S is a variety of dimension ≤ 3 as $r > 3$.

Let $C := \{(x, y, z) \in X \times X \times \mathbb{P}^r$: if $x \neq y$, then $z \in \overline{xy}$; if $x = y$, then $z \in E^*_{x,X}\}$. C is the closure of $C \cap ((X \times X - \Delta) \times \mathbb{P}^r)$ in the classical topology: if x_n , $y_n \to x$ as $n \to \infty$, then $\overline{x_n y_n} \to E^*_{x,X}$. Thus C is algebraic. Let $\pi_{12} : C \to X \times X$ and $\pi_3 : C \to \mathbb{P}^r$ be the natural projections. By definition of C , the fibres of π_{12} are lines, so irreducible 1-dimensional. As $X \times X$ is irreducible 2-dimensional, then C is irreducible 3-dimensional. As $S = p_3(C)$, then S is irreducible and $\leq$ 3-dimensional.

Step 1b: Projecting X to $\mathbb{P}^2$

By Step 1a, we may assume $X \subseteq \mathbb{P}^3$.

Definition : For $x, y \in X$, $\overline{xy}$ is a tri-secant of X if $\overline{xy}$ meets X

in a third point or $\overline{xy}$ is tangent to X at x or y .

To prove Step 1b, It suffices to find $x \in \mathbb{P}^3$ s.th.

i) $x \notin \bigcup_{u \in X} E^*_{u,X}$;

ii) x is in only finitely many secants $\overline{u_i v_i}$ of X , $1 \leq i \leq \nu$;

iii) $\forall i$, $\overline{u_i v_i}$ is not a tri-secant of X ;

iv) $\forall i$, the planes $\overline{x\ , E^*_{u_i,X}}$ and $\overline{x\ , E^*_{v_i,X}}$ are distinct.

If so, then ii) $\Rightarrow p_x(X)$ is birational to X and i) $\Rightarrow$ its only singular points are $w_i := p_x(u_i) = p_x(v_i)$. By i) and iii), we have that the multiplicity of w_i on $p_x(X)$ is 2. Finally by iv), these two branches have distinct tangent lines, namely $p_x(E^*_{u_i,X})$ and $p_x(E^*_{v_i,X})$. Thus w_i is an ordinary double point. To show such an x exists, consider $T := \{(x,y) \in X \times X - \Delta : \overline{xy}$ meets X in a third point or $\overline{xy}$ is tangent to X at x or $y\}$ and $B := \{(x,y) \in X \times X - \Delta : E^*_{x,X}$ and $E^*_{y,X}$ lie in a plane$\}$.

Showing $T \cup B \subsetneq (X \times X - \Delta)$

T is the closure of $\{(x, y) \in X \times X - \Delta : \overline{xy}$ meets X in a third point$\}$ in $X \times X - \Delta$ and coplanarity is a closed property so T and B are algebraic. Let $x \in X$ and $l := E^*_{x,X}$. Consider the projection $p_l : X \to \mathbb{P}^1$. Let $\alpha \in \mathbb{P}^1$ be a point where p_l is smooth and let $L := p_l^{-1}(\alpha) \cup l$, $L \cap X = \{x, y_1, ..., y_k\}$ by Noether Normalization. If $y \notin l$ and p_l is smooth at y , then by dimension, we have $p_l(E^*_{y,X}) = \mathbb{P}^1$. As $y_i \notin l \ \forall i$, then $E^*_{y_i,X} \not\subseteq L$ so $(x, y_i) \notin B$. Let $\phi : \{z : |z| < \epsilon\} \to X$, $\phi(0) = x$, be a chart on X near x .

Consider secants $\overline{y_1\phi(z)}$ as z varies. Since $(x, y_1) \notin B$, then $(\phi(z), y_1) \notin B$ for $|z|$ small. If $(\phi(z), y_1) \in T$ for all $|z|$ small, then $\exists i$, $2 \leq i \leq k$ and sequences $z_n \to 0$, $y_i^{(n)} \to y_i$, $y_i^{(n)} \in X$, as $n \to \infty$ s.th. $\phi(z_n), y_1, y_i^{(n)}$ are collinear. Then $\overline{y_i y_i^{(n)}}$ would be in the plane $\overline{y_1 x \phi(z_n)}$. Taking $n \to \infty$, this plane approaches the join of y_1 and $l = \lim \overline{x\phi(z_n)}$, which is L .

Thus the line $E_{y_i,X} = \lim \overline{y_i^{(n)}, y_i}$ lies in L , contradicting the fact that $(x, y_i) \notin B$. Thus $(\phi(z), y_1) \in X \times X - \Delta - B - T$ for $|z|$ small. Let C be as in Step 1a, $C^* := C \cap [(\Delta \cup B \cup T) \times \mathbb{P}^3]$, and $S^* := \pi_3(C^*)$. Then C^* is algebraic with all components having dimension ≤ 2 , and so the same for S^* . Thus choose $x \notin S^*$ to satisfy i) - iv).

Inflexion Points

Definition : If $x \in X$ is a smooth point, we say x is an inflexion point if $i(x; X \cap E^*_{x,X}) \geq 3$.

Proposition 1 : 1) If $f(X_0, X_1, X_2) = 0$ is the equation of X , then $\{\text{inflexion points}\} = (X - \text{Sing } X) \cap \{\text{zeroes of } H = \det(\frac{\partial^2 f}{\partial X_i \partial X_j})\}$;

2) If X is described analytically in affine coordinates near a point P by $X_2 = f(X_1)$, then $\{\text{inflexion points near } P\} = \{\text{points } (a, f(a)) \text{ where } f''(a) = 0\}$.

Proof : 1) As the Hessian transforms like a quadratic form under change of coordinates, the zeroes of H are unchanged. Then choose coordinates s.th. $x = (1, 0, 0) \in X$ is a smooth point and $E^*_{x,X} = V(X_1)$. Then $f = \alpha X_1 X_0^{d-1} + \beta X_2^2 X_0^{d-2} + \gamma X_1 X_2 X_0^{d-2} + \delta X_1^2 X_0^{d-2} +$ (terms with X_1, X_2 to powers > 2) as $f(x) = 0$. Then $H(x) = -2(d-1)^2\alpha^2\beta$. If $\alpha = 0$, then the Jacobian of f at x is zero, but x is a smooth point. Thus $\alpha \neq 0$ so $H(x) = 0$ iff $\beta = 0$.

We have $X \cap E^*_{x,X}$ = zeroes of $f(X_0, 0, X_2)$ = zeroes of $\beta X_2^2 X_0^{d-2}$

+(terms with X_2 to power > 2). Thus $\beta = 0$ iff $i(x; X \cap E^*_{x,X}) > 2$.

2) This is a fact in undergraduate calculus. ■

Corollary 2 : If $\deg X > 1$, then X has only finitely many inflexion points.

Proof : Since X is not a line, $f'' \not\equiv 0$ so by 2), not every point is an inflexion point. By 1), the set of inflexion points is algebraic, so its components have dimension $< \dim X = 1$. ■

Step 1c: Projection X to $\mathbb{P}^1$

By Step 1b, we may assume $X \subseteq \mathbb{P}^2$ is smooth except for finitely many ordinary double points. Choose $x \in \mathbb{P}^2$ s.th. $x \notin \bigcup_{\{y \in X : y \text{ a point of inflexion or a singular point}\}} E^*_{y,X} \cup X$. Consider $p_x : X \to \mathbb{P}^1$. If $y \in X$ is smooth and $x \notin E^*_{y,X}$, then p_x is smooth at y. If y is a double point, by choice of x , p_x is smooth on each branch. If y is smooth and $x \in E^*_{y,X}$, y is not a point of inflexion. Thus $2 = i(y; X \cap E^*_{y,X}) = i(y; X \cap p_x^{-1}(p_x(y))) = \mathrm{mult}_y(\mathrm{res}_X \, p_x)$. Hence y is an ordinary branch point. By Cor. 5.6, $\deg p_1 = d$.

Step 2a: $g_a(X) = \frac{1}{2}(d-1)(d-2) - \delta$

By Step 1, we have $p_2 : X \to X' := p_2(X) = V(F)$ for some polynomial F and X' is smooth everywhere except at the w_i , $1 \leq i \leq \delta$. Then $X - \{u_1, v_1, ..., u_\delta, v_\delta\}$ and $X' - \{w_1, ..., w_\delta\}$ correspond biregularly. Thus for $x \in X - \{u_1, ..., v_\delta\}$ and $x' := p_2(x)$, we can say $\mathcal{O}_{x,X} = \mathcal{O}_{x',X'}$ by identifying $\mathbb{C}(X)$ with $\mathbb{C}(X')$.

Local Ring of w_i

Lemma 3 : $\mathcal{O}_{w_i,X'} = \{\alpha \in \mathcal{O}_{u_i,X} \cap \mathcal{O}_{v_i,X} : \alpha(u_i) = \alpha(v_i)\}$

Proof : Let Y, Z be affine coordinates in $\mathbb{P}^2$ s.th. $w_i = (0,0)$ and the affine equation of F is of the form $F = YZ$ + higher order terms. This is possible as the branches meet transversely. Recall that $\forall n$, $\forall f \in \mathcal{O}_{w_i,\mathbb{P}^2}$, we have an expansion $f = \sum_{i+j<n} c_{ij} Y^i Z^j$ + (remainder in $\mathfrak{M}^n_{w_i,\mathbb{P}^2}$) $(*)$ where $\mathfrak{M}_{w_i,\mathbb{P}^2}$ is the maximal ideal in $\mathcal{O}_{w_i,\mathbb{P}^2}$. Modulo F , we then have $f = a + \sum_{i=1}^{n-1} b_i Y^i$ + $\sum_{i=1}^{n-1} c_i Z^i$ + (remainder in $\mathfrak{M}^n_{w_i,\mathbb{P}^2}$).

As an analytic set near w_i, X' is the union of 2 smooth branches with tangent lines $Y = 0$ and $Z = 0$. On the branch with tangent line $Y = 0$, Z vanishes to 1st order and Y to higher order. Suppose this branch corresponds to a neighborhood of u_i on X . Then $Z \circ p_2$ vanishes to 1st order at u_i and $Y \circ p_2$ vanishes to higher order. The opposite at v_i . Thus $\forall f \in \mathfrak{M}_{u_i,X} \cap \mathfrak{M}_{v_i,X}$, we have an expansion $f = \sum_{i=1}^{n-1} b_i Y^i + \sum_{i=1}^{n-1} c_i Z^i +$ (remainder vanishing to order n at u_i and v_i) $(**)$.

Let $f \in \mathcal{O}_{u_i,X} \cap \mathcal{O}_{v_i,X}$ s.th. $f(u_i) = f(v_i) = 0$. Write $f = g/h$ where $g, h \in \mathcal{O}_{w_i,X'}$. Expanding f as in $(**)$, we have $f = f_n(Y, Z) + R_n$ where f_n is a polynomial of degree $n-1$ and R_n vanishes to order n at u_i and v_i. Then $g = hf_n + hR_n$. As $g, hf_n \in \mathcal{O}_{w_i,X'}$, so is hR_n . Expanding hR_n as in $(*)$, we have $a = b_i = c_i = 0 \ \forall i$ else hR_n would not vanish to order n at u_i and v_i . Thus $hR_n \in \mathfrak{M}^n_{w_i,X'}$. Thus $\forall n$, $g \in h\mathcal{O}_{w_i,X'} + \mathfrak{M}^n_{w_i,X'}$. Then by Krull, $g \in h\mathcal{O}_{w_i,X'}$ so $f \in \mathcal{O}_{w_i,X'}$. The other direction is by definition. ■

Continuing Proof of Step 2a

Consider $R' := \mathbb{C}[X_0, X_1, X_2]/(F) \subseteq R := \mathbb{C}[X_0, ..., X_r]/I(X)$.

WLOG, assume $X_i(w_j) \neq 0, i = 0, 1, 2, j = 1, ..., \delta$. $\forall k$, consider the sequence $0 \to R'_k \xrightarrow{\iota} R_k \xrightarrow{\alpha} \sum_{i=1}^{\delta} \mathbb{C} \to 0$ where $\alpha(f) = (..., \frac{f}{X_0^k}(u_i) - \frac{f}{X_0^k}(v_i), ...)$. By Lemma 3, $\alpha \circ \iota = 0$. If $k \gg 0$, $\exists$ hypersurfaces $H_i = V(G_i)$ s.th. $u_j, v_j \in H_i \ \forall j \neq i$, $u_i \notin H_i$, and $v_i \in H_i$. Then $\alpha(G_i) = c \cdot (i\text{th unit vector})$, so α is surjective. As p_2 is birational, R and R' have the same fraction field. Then by the next lemma and Prop. 6.11, the sequence will be exact.

Lemma 4 : $\forall k$ and $\forall G \in R_k$ s.th. $\alpha(G) = 0$, $\exists n$ s.th. $X_0^n G, X_1^n G, X_2^n G \in R'$.

Proof : It suffices to show $X_0^n G \in R'$ by symmetry. Consider the affine rings $S' := \mathbb{C}[\frac{X_1}{X_0}, \frac{X_2}{X_0}]/(\frac{F}{X_0^d}) \subseteq S := \mathbb{C}[\frac{X_1}{X_0}, ..., \frac{X_r}{X_0}]/I(X)$. Then it suffices to show if $g \in S$ is s.th. $g(u_i) = g(v_i)$ $\forall i$, then $g \in S'$. By Proposition 1.11, $S = \bigcap_{\{x \in X : x \notin V(X_0)\}} \mathcal{O}_{x,X}$ and $S' = \bigcap_{\{x \in X' : x \notin V(X_0)\}} \mathcal{O}_{x,X'}$. Then the result follows from Lemma 3 and biregularity of p_2 . $\blacksquare$

Thus we have the sequence to be exact for $k \gg 0$. We have

$P_X(k) = \dim R_k$

$= \dim R'_k + \dim \sum_{i=1}^{\delta} \mathbb{C}$

$= \dim[\mathbb{C}[X_0, X_1, X_2]/(F)]_k + \delta$

$= \dim \mathbb{C}[X_0, X_1, X_2]_k - \dim F \cdot \mathbb{C}[X_0, X_1, X_2]_{k-d} + \delta$

$= \binom{k+2}{2} - \binom{k-d+2}{2} + \delta$

$= kd + 1 - \frac{1}{2}(d-1)(d-2) + \delta.$

Thus $g_a(X) = \frac{1}{2}(d-1)(d-2) - \delta.$

Step 2b: $d(d-1) = \beta + 2\delta$

Let X_0, X_1, X_2 be the coordinates of $\mathbb{P}^2$, X_0, X_1 be the coordinates in $\mathbb{P}^1$ and $p_x : \mathbb{P}^2 - \{x\} \to \mathbb{P}^1$ be the projection. Let F be the equation of $p_2(X)$. Consider the curve Y defined by $\partial F / \partial X_0 = 0$. As $\deg Y = d-1$, by Bezout, $\deg(Y \cdot p_2(X)) = d(d-1)$. To prove $d(d-1) = 2\delta + \beta$, it suffices to show:

i) $Y \cap p_2(X) =$ (double points of $p_2(X)$ and branch points of p_x) ;

ii) at each double point y , $i(y; Y \cap p_2(X)) = 2$;

iii) at each branch point y , $i(y; Y \cap p_2(X)) = 1$.

Step 2bi)

We have $p_2(X)$ covered by affine pieces $X_1 \neq 0$ and $X_2 \neq 0$ so look in the first piece and let $u = X_0/X_1, v = X_2/X_1$ be affine coordinates. Then p_x is the projection of the (u, v)-plane to the u-line, $p_2(X)$ has affine equation $f(u, v) := F(u, 1, v)$ and Y has affine equation $\frac{\partial F}{\partial X_0}(u, 1, v) = \partial f/\partial u$. Then $\forall y \in p_2(X)$, $\frac{\partial f}{\partial u}(y)$ $\neq 0$ iff y is a smooth point of $p_2(X)$ and $E^*_{y,p_2(X)}$ projects onto the u-line so i) follows.

Step 2bii)

If y is a double point and coordinates (u, v) are chosen with $y = (0, 0)$, then because neither branch of $p_2(X)$ at y is parallel to the v-axis, $f = (au + bv)(cu + dv) + \deg \geq 3$ terms, $ad - bc \neq 0$, $a \neq 0$, $c \neq 0$ so $\frac{\partial f}{\partial u} = 2acu + (ad + bc)v + (\deg \geq 2 \text{ terms})$. As $ad - bc \neq 0$, we have $\frac{ad+bc}{2ac} \neq b/a$ or d/c. Then y is a smooth point for Y with tangent line unequal to either tangent line to $p_2(X)$ at y so ii) follows.

Step 2biii)

If y is a branch point of $p_x : p_2(X) \to \mathbb{P}^1$, and coordinates (u, v) are chosen s.th. $y = (0, 0)$, then as y is smooth and the branching is ordinary, we have $f = av + bu^2 + cuv + dv^2 + (\deg \geq 3 \text{ terms})$, $a \neq 0$, $b \neq 0$ so $\frac{\partial f}{\partial u} = 2bu + cv + (\deg \geq 2 \text{ terms})$. Thus y is a smooth point of Y . Also, Y and $p_2(X)$ meet transversely at y , proving iii).

Step 2c: $e(X) = 2d - \beta$

Consider the covering $p_1 : X \to \mathbb{P}^1$. Let $x_1, ..., x_\beta \in X$ be the branch points of p_1 and let $t_i = p_1(x_i)$. Triangulate $\mathbb{P}^1$ s.th. the t_i are vertices. Take all points of X over vertices of $\mathbb{P}^1$ to be vertices of X. For all edges $f : \Delta^1 \to \mathbb{P}^1$ in the triangulation, the covering X is unramified over $f(\mathrm{Int}(\Delta^1))$. Since $\mathrm{Int}(\Delta^1)$ is simply connected, f lifts to d distinct maps $f_0^{(i)} : \mathrm{Int}(\Delta^1) \to X$ with disjoint images. As $p_1 : X \to \mathbb{P}^1$ is proper, we can extend the $f_0^{(i)}$ to maps $f^{(i)} : \Delta^1 \to X$ lifting f .

Let these be the edges of X and repeat the process for faces to have a triangulation of X. Suppose $\mathbb{P}^1$ has s_0 vertices, s_1 edges, and s_2 faces, so X has ds_1 edges and ds_2 faces. Among the ds_0 potential vertices of X over the t_i , β are branch points so there are $ds_0 - \beta$ vertices. Thus

$$e(X) = (ds_0 - \beta) - (ds_1) + (ds_2)$$

$$= d(s_0 - s_1 + s_2) - \beta$$

$$= d(e(\mathbb{P}^1)) - \beta = 2d - \beta.$$