Explicit Inverse Mapping Theorem

Siu Sing Chow

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1. Classical Inverse Mapping Theorem

Suppose $D \subset \mathbb{R}^n$ open, $\phi \in C^1(D, \mathbb{R}^n)$, $a \in D$ and $\det(D\phi(a)) \neq 0$.

$\exists \phi^{-1}$ at $\phi(a)$, but for what $r > 0$ $\exists \psi := \phi^{-1} \in C^1(B_r(\phi(a)), D)$,

where $B_r(b) := \{y : \|y - b\| \leq r\}$

Say $\delta \in C^0(D, \mathbb{R})$ and $T \in C^0(D, lin(\mathbb{R}^{n \times n}))$ such that

$\frac{\partial \phi}{\partial x} \cdot T \equiv \delta \cdot Id$, e.g. $\delta(x) := \det\left(\frac{\partial \phi}{\partial x}\right) \Rightarrow T(x) := \left(\frac{\partial \phi}{\partial x}\right)^{\#}$

2. Notations and Tools

$$\nabla\phi(x,y) := \int_0^1 \frac{\partial\phi}{\partial x}(tx+(1-t)y)\,dt$$

$$\phi_a(x) := -\frac{1}{2}\left(\frac{\partial\phi}{\partial x}(a)\right)^{-1}(\phi(x)-\phi(a))+\phi(a)$$

$$\Longrightarrow \quad \phi_a(a)=\phi(a) \qquad \frac{\partial\phi_a}{\partial x}(a) = -\frac{1}{2}Id$$

$$\nabla\phi_a(x,y) = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x}(a)\right)^{-1}\nabla\phi(x,y) = -\frac{1}{2}\frac{1}{\delta(a)}\left(\frac{\partial\phi}{\partial x}(a)\right)^{\#}\nabla\phi(x,y)$$

$$\nabla\phi(x,y)(x-y)=\phi(x)-\phi(y) \;,\; \nabla\phi_a(x,y)(x-y)=\phi_a(x)-\phi_a(y)$$

3. Explicit Inverse Mapping Theorem

$$d\phi \in Lip\left(D, \mathbb{R}^n\right),\ K \subset D \text{ compact},\ C_1 = \min_{a \in K} \frac{1}{2} \left\| \left(\frac{\partial \phi}{\partial x}\left(a\right) \right)^{\#} \right\|^{-1} > 0.$$

$$C_0 > 0 \text{ s.th } C_0 \cdot \sup_{a \in K,\, |z| \le C_0 |\delta(a)|} \frac{1}{|z|} \left\| \frac{\partial \phi}{\partial x}\left(a + z\right) - \frac{\partial \phi}{\partial x}\left(a\right) \right\| \le \frac{1}{2} C_1$$

Theorem: $\forall a \in K$, $0 \le c \le C_0$ and $r := \left|\delta\left(a\right)\right|$,

I. $\phi\left(B_{cr}\left(a\right)\right) \supset B_{C_1 cr^2}\left(\phi\left(a\right)\right)$

II. $\phi : B_{cr}\left(a\right) \to \phi\left(B_{cr}\left(a\right)\right)$ is 1:1.

Cor: $\exists \phi^{-1} \in C^1\left(B_{C_1 cr^2}\left(\phi\left(a\right)\right), B_{cr}\left(a\right)\right)$.

4. Claims $r(a) = \sup\left\{r \middle| \left(a, B_r(a), B_r(a)\right) \subset Z\right\}$,

$$Z := \left\{(a,x,y) \mid \|\nabla\phi_a(x,y)\| \le \tfrac{3}{4}\right\} \cap \left\{(a,x,y) \mid \|Id + \nabla\phi_a(x,y)\| \le \tfrac{3}{4}\right\}$$

$$= \left\{(a,x,y) \mid \left\| \left(\frac{\partial\phi}{\partial x}(a)\right)^{\#} \nabla\phi(x,y) \right\| \le \frac{3}{2}|\delta(a)| \right\} \cap$$

$$\left\{(a,x,y) \mid \left\| \delta(a)\, Id - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}(a)\right)^{\#} \nabla\phi(x,y) \right\| \le \frac{3}{4}|\delta(a)| \right\}$$

$$Claim1: \quad r(a) \ge C_0 |\delta(a)|$$

$$Claim2: \quad \forall c : 0 \le c \le 1 \quad \Rightarrow \quad a.\ \ \phi_a : B_{cr(a)}(a) \to B_{cr(a)}(\phi(a)) \ \ is\ 1:1$$

$$b.\ \ \phi_a\left(B_{cr(a)}(a)\right) \supset B_{\frac{c}{4}r(a)}(\phi(a))$$

5. Claims $\Rightarrow$ Theorem

$$\phi(x) - \phi(a) = -2\left(\frac{\partial \phi}{\partial x}(a)\right)(\phi_a(x) - \phi_a(a)), \text{ and}$$

$$|\phi(x) - \phi(a)| \leq cr(a)\,|\delta(a)|\,C_1 \Rightarrow |\phi_a(x) - \phi_a(a)|$$

$$\leq \frac{1}{2}\left\|\left(\frac{\partial \phi}{\partial x}(a)\right)\right\|^{-1} |\phi(x) - \phi(a)|$$

$$\leq \frac{1}{2}\left\|\left(\frac{\partial \phi}{\partial x}(a)\right)\right\|^{-1} cr(a)\,|\delta(a)|\,C_1 \leq \frac{1}{2}\left\|\left(\frac{\partial \phi}{\partial x}(a)\right)^{-1}\right\| cr(a)\,|\delta(a)|\,C_1$$

$$\leq \frac{1}{4}cr(a) \text{ since } \left\|\left(\frac{\partial \phi}{\partial x}(a)\right)^{-1}\right\| |\delta(a)|\,C_1 \leq \frac{1}{2}$$

6. Claims $\Rightarrow$ Theorem

i.e. $\phi_a(x) \in B_{\frac{\epsilon}{4}r(a)}(\phi_a(a)) \overset{(2b)}{\subset} \phi_a\left(B_{cr(a)}(a)\right)$

$\overset{(2a)}{\Rightarrow} x \in B_{cr(a)}(a) \Rightarrow \phi(x) \in \phi\left(B_{cr(a)}(a)\right)$

Summarizing: $\phi\left(B_{cr(a)}(a)\right) \supset B_{cr(a)|\delta(a)|C_1}(\phi(a))$

With Claim 1 and $cr(a) = c'|\delta(a)| \Rightarrow$(I), with $0 \leq c' \leq C_0$

$\forall x, y \in B_{cr(a)}(a)$, $\phi_a(x) = \phi_a(y) \overset{(2a)}{\Rightarrow} x = y$

$\Longleftrightarrow \phi(x) = \phi(y) \Rightarrow x = y \Rightarrow$(II)

7. Claim 1: $\max\{|x|,|y|\} \leq C_0 |\delta(a)|$, $z := tx + (1-t)y$

$$\left\| \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} (\nabla\phi(a+x,a+y) - \nabla\phi(a,a)) \right\| \cdot \left\| \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \right\|^{-1}$$

$$\leq \|(\nabla\phi(a+x,a+y) - \nabla\phi(a,a))\|$$

$$\leq \left\| \int_0^1 \frac{\partial \phi}{\partial x}(t(a+x) + (1-t)(a+y)) - \frac{\partial \phi}{\partial x}(a)\, dt \right\|$$

$$\leq \left\| \int_0^1 \frac{\partial \phi}{\partial x}(a+z) - \frac{\partial \phi}{\partial x}(a)\, dt \right\| \leq \frac{|z|}{2} \frac{C_1}{C_0}$$

$$\leq |\delta(a)| \frac{1}{2} C_1 \leq \frac{1}{4} |\delta(a)| \left\| \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \right\|^{-1}$$

Combine with, $\left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \nabla \phi(a, a) = \delta(a) \, Id \Rightarrow$

$$\left\| \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \nabla \phi(a+x, a+y) - \delta(a) \, Id \right\| \leq \frac{1}{4} |\delta(a)|$$

$$\Rightarrow \left\| \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \nabla \phi(a+x, a+y) \right\| \leq \frac{5}{4} |\delta(a)| \qquad (*) \ .$$

Moreover, $\left\| \frac{1}{2} \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \nabla \phi(a+x, a+y) - \frac{1}{2} \delta(a) \, Id \right\| \leq \frac{1}{8} |\delta(a)|$

$$\Rightarrow \left\| \delta(a) \, Id - \frac{1}{2} \left(\frac{\partial \phi}{\partial x}(a)\right)^{\#} \nabla \phi(a+x, a+y) \right\| \leq \frac{5}{8} |\delta(a)| \qquad (**) \ .$$

Now $\max\{|x|, |y|\} \leq C_0 |\delta(a)|$, (*) and (**)$\Rightarrow (a, a+x, a+y) \in Z \Rightarrow$

$r(a) \geq C_0 |\delta(a)|$, as required.

9. Proof of Claim 2a:

For $x \in B_{cr(a)}(a)$, $|\phi_a(x) - \phi(a)| = |\nabla\phi_a(x,a)(x-a)|$

$\leq \|\nabla\phi_a(x,a)\| \, |x-a| \leq \frac{3}{4} cr(a) \leq cr(a) \Rightarrow$

$\phi_a\left(B_{cr(a)}(a)\right) \subset B_{cr(a)}(\phi(a))$. Say $x, y \in B_{cr(a)}(a)$ and $\phi_a(x) = \phi_a(y)$

$\Rightarrow \nabla\phi_a(x,y)(x-y) = 0 \Rightarrow (Id + \nabla\phi_a(x,y))(x-y) = x-y$

$\Rightarrow |x-y| = \|Id + \nabla\phi_a(x,y)\| \, |x-y| \Rightarrow |x-y| \leq \frac{3}{4}|x-y| \Rightarrow x = y \; \square$

10. Proof of Claim 2b:

Let $b \in B_{\frac{c}{4}r(a)}\left(\phi\left(a\right)\right)$ and set $f_b\left(x\right) := x + \phi_a\left(x+a\right) - b$.

Then, $f_b\left(x\right) - f_b\left(y\right) = \left(Id + \nabla\phi_a\left(x+a, y+a\right)\right)\left(x-y\right)$.

For $x \in B_{cr(a)}\left(0\right)$, $\left|f_b\left(x\right)\right| \leq \left|x + \phi_a\left(x+a\right) - \phi_a\left(a\right)\right| + \left|\phi\left(a\right) - b\right|$

$\leq \left\|Id + \nabla\phi_a\left(x+a, a\right)\right\| \left|x\right| + \frac{1}{4}cr\left(a\right) \leq \frac{3}{4}cr\left(a\right) + \frac{c}{4}r\left(a\right) = cr\left(a\right)$

So, $f_b\left(B_{cr(a)}\left(0\right)\right) \subset B_{cr(a)}\left(0\right)$.

11. Proof of Claim 2b:

For $x, y \in B_{cr(a)}(0)$,

$$|f_b(x) - f_b(y)| = |(I + \nabla\phi_a(x+a, y+a))(x-y)| \le \frac{3}{4}|x-y|,$$

Let $x^0 = 0,\ x^k := f_b\left(x^{k-1}\right)$ for $k \ge 1$.

$\exists x^* \in B_{cr(a)}(0)$ s.t. $x^k \to x^*$ and $f_b(x^*) = x^*$

It implies $\phi_a(x^* + a) = b$ $\square$

12. Example

Say $\phi : M \to K = \phi(M) \Subset \mathbb{R}^m$.

$J_\phi(x) \approx x_1^{\alpha_1} \dots x_m^{\alpha_m} =: x^\alpha$ locally. Say $g(y) := dist(y, \mathbb{R}^m \backslash K)$.

Question: Estimate $1 \leq s < \infty$ s.th.

$\exists \epsilon_0 > 0$, $\forall b \in \partial K$, $\forall 0 \leq t \leq \epsilon_0$

$\exists y : t = |y - b| \leq \epsilon_0$ and $|g(y)| \gtrsim |y - b|^s = t^s > 0$

Claim: $s = \max 2|\alpha|$.

13. Proof

$a = 0,\ b = \phi(0)$. Locally $g(\phi(x)) \gtrsim |J_\phi(x)|^2 = |x^\alpha|^2$,

by Exp. Inv. Mapping Thm. $|g(y)| = |g(y) - g(b)|$

$\gtrsim |x - a|^{2|\alpha|} \gtrsim |\phi(x) - \phi(a)|^{2|\alpha|} = |y - b|^{2|\alpha|}$, provided $y := \phi(x)$

with $|x - a|^{|\alpha|} \cong |x^\alpha|$, e.g. when $|x_1| = \ldots = |x_1|$, and since

$|\phi(x) - \phi(a)| \leq c\,|x - a|$