Pierre-Grant's Chow-type Theorem for Coherent Ideals.

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Introduction: Coherent Complex Analytic Ideals $\mathcal{I}$

refers to ideals in holomorphic functions f , shortly $f \in \mathcal{H}$, on open U in $\mathbb{C}$-analytic manifolds M and coherent means that the germs of functions of $\mathcal{I}$ at a in M (shortly stalks $\mathcal{I}_a$ and de facto finitely generated, say by $\{f_j\}_{1\leq j\leq l}$, ideals in the rings of convergent power series $\mathcal{O}_{M,a}$ on M at a) generate by means of $\{f_j\}$ ideals $\mathcal{I}_b \hookrightarrow \mathcal{O}_{M,b}$ for all b nearby a .

Theorem: U is a nbhd of $0 \in \mathbb{C}^r$, $M := U \times \mathbb{CP}^n$, $\mathcal{I}$ coherent $\Rightarrow$ $\mathcal{I}$ is **relatively algebraic**, i.e. after shrinking U it is generated by finitely many homogeneous polynomials in $\mathbb{CP}^n$-coordinates with coefficients in $\mathcal{H}(U)$.

Remark: Chow Thm for $X \hookrightarrow \mathbb{CP}^n$ follows with U a singleton and $\mathcal{I}_X$ in $\mathcal{O}_{\mathbb{CP}^n}$ with sections $\mathcal{S}(\mathcal{I}_X)(V)$ over nbhds V being the ideals of f in $\mathcal{S}(\mathcal{O}_{\mathbb{CP}^n})(V)$ vanishing on $X \cap V$, since $\mathcal{I}_X$ is coherent by **Oka's Thm**.

For proper $\mathbb{C}$-analytic maps $F : M \to N$ **of manifolds** (as below !) coherency of $\mathcal{I} \hookrightarrow \mathcal{O}_N$ implies the coherency of the **pull back** $F^*\mathcal{I} \hookrightarrow \mathcal{O}_M$ whose sections over $F^{-1}(V)$ for nbhds V are generated by $f \circ F$, $f \in \mathcal{S}(\mathcal{I})(V)$. Let $x = (x_1, \ldots, x_r)$ be coordinates on U .

Fact 1. For proper maps F as above coherency of $\mathcal{I} \hookrightarrow \mathcal{O}_M$ implies the coherency of the **direct image** $F_*\mathcal{I}$ whose sections $\mathcal{S}(F_*\mathcal{I})(V) := \mathcal{S}(\mathcal{I})(F^{-1}(V))$ on nbds V and $F_*\mathcal{I} \hookrightarrow \mathcal{O}_N$ for **blowings up**, e.g. for **blow-up** $\tilde{\mathbb{C}}^{n+1} := \{y_i\xi_j = y_j\xi_i\} \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{CP}^n$ of $\mathbb{C}^{n+1}$ at its 0 , where $y = (y_0, \ldots, y_n)$ and homogeneous $[\xi] = [\xi_0 : \ldots : \xi_n]$ are coordinates on $\mathbb{C}^{n+1}$ and $\mathbb{CP}^n$ respectively. Holomorphic maps $\pi_1 : \tilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ and $\pi_2 : \tilde{\mathbb{C}}^{n+1} \to \mathbb{P}^n$ are the restrictions to $\tilde{\mathbb{C}}^{n+1}$ of projections of $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ to $\mathbb{C}^{n+1}$ and, respectively, to $\mathbb{CP}^n$. Let maps $\sigma_j := \mathrm{id}_U \times \pi_j$, $j = 1, 2$.

Note: $\{(y,[\xi]) : \xi_j = 1, y_k = y_j\xi_k \ \forall k \neq j\} = \tilde{\mathbb{C}}^{n+1} \cap \{\xi_j \neq 0\} \cong \mathbb{C}^{n+1}$.

For our coherent ideal $\mathcal{I} \hookrightarrow \mathcal{O}_{U\times\mathbb{CP}^n}$ ideal $\tilde{\mathcal{I}} := \sigma_2^*(\mathcal{I}) \hookrightarrow \mathcal{O}_{U\times\tilde{\mathbb{C}}^{n+1}}$ is generated by the restrictions to $U \times \tilde{\mathbb{C}}^{n+1}$ of sections of $\mathcal{I}$ considered as functions on $U \times \mathbb{C}^{n+1} \times \mathbb{P}^n$ constant along $\mathbb{C}^{n+1}$ (and is coherent, as well as the direct image $\mathcal{J} := (\sigma_1)_*(\tilde{\mathcal{I}}) \hookrightarrow \mathcal{O}_{U\times\mathbb{C}^{n+1}}$ and $\tilde{\mathcal{J}} := \sigma_1^*(\mathcal{J})$).

Note: $\tilde{\mathcal{J}} \subset \tilde{\mathcal{I}}$ and $\tilde{\mathcal{J}} = \tilde{\mathcal{I}}$ off $\sigma_1^{-1}(U \times \{0\})$ where σ_1 is biholomorphic.

Fact 2. $\mathbb{C}$-analytic Nullstellensatz: For any G in the stalk $\tilde{\mathcal{I}}_q$, $q \in \sigma_1^{-1}(U \times \{0\}) =: \{z = 0\}$, exists integer $d > 0$ s.th. $z^d \cdot G \in \tilde{\mathcal{J}}_q$.

Plan: Show stalk $\mathcal{J}_{(0,0)} \hookrightarrow \mathbb{C}\{x, y\}$ is generated by $P_j \in \mathbb{C}\{x\}[y]$ homog. in y . **Then that** with $y := [\xi]$ the latter generate $\mathcal{I}$ near $\{0\} \times \mathbb{CP}^n$.

Lemma 1. $F \in \mathcal{J}_{(0,0)}$, $F^{(\lambda)} := F(x, \lambda y) \Rightarrow F^{(\lambda)} \in \mathcal{J}_{(0,0)}$, $\forall\ \lambda \in \mathbb{C}^*$.

Proof. Let $H \in \mathcal{H}$ in a nbhd of $(0, 0)$, then $H \in \mathcal{J}_{(0,0)}$ iff $\sigma_1^* H$ is a section of $\tilde{\mathcal{I}}$ over some nbhd of $\sigma_1^{-1}(0, 0) = \{0\} \times \mathbb{CP}^n$ iff $\sigma_1^* H \in (\sigma_2^* \mathcal{I})_p$, $\forall\ p \in \sigma_1^{-1}(0, 0)$, due to the def. of $(\sigma_1)_*$. Let $p \in \sigma_1^{-1}(0, 0)$, $q := \sigma_2(p)$ and coordinates $[\xi]$ on $\mathbb{CP}^n$ s.th. $q = (0, [1, 0, \ldots, 0])$. Let $W := \{\xi_0 \neq 0\}$, then $w_i := \xi_i / \xi_0$ are nonhomogeneous coordinates on it.

Then, $\sigma_2^{-1}(U \times W) \cong U \times \mathbb{C} \times W$ is a nbhd of p in $U \times \tilde{\mathbb{C}}^{n+1}$ with coordinates (x, y_0, w) , $\sigma_1(x, y_0, w) = (x, y_0, y_0 \cdot w)$ and $\sigma_2(x, y_0, w) = (x, w)$. Coherency of $\mathcal{I} \Rightarrow \exists \{G_j\}$ generating $\mathcal{I}$ over a nbhd of $q \Rightarrow \{\sigma_2^* G_j\}$ generate $\tilde{\mathcal{I}}$ over nbhd of $p = (0, 0, 0)$. Since $\sigma_1^* F \in \tilde{\mathcal{J}} \subset \tilde{\mathcal{I}} \Rightarrow \exists \{A_j\} \subset \mathcal{H}$ on a nbhd of p s.th. $\sigma_1^* F(x, y_0, w) = \sum_j A_j \cdot \sigma_2^* G_j$. For $\lambda \in \mathbb{C}^*$ and y_0 small enough it follows $\sum_j A_j(x, \lambda y_0, w)\sigma_2^* G_j(x, y_0, w) = \sum_j A_j(x, \lambda y_0, w) G_j(x, w) = \sum_j A_j(x, \lambda y_0, w)\sigma_2^* G_j(x, \lambda y_0, w) = \sigma_1^* F^{(\lambda)}(x, y_0, w)$. **So** $\sigma_1^* F^{(\lambda)} \in (\tilde{\mathcal{I}})_p$, $\forall\, p \in \sigma_1^{-1}(0, 0)$ and $F^{(\lambda)} \in \mathcal{J}_{(0,0)}$ ■

For F holomorphic on a nbhd of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ write:

$F(x,y) =: \sum_k \sum_{|\alpha|=k} a_\alpha(x) y^\alpha =: \sum_k F_k(x,y)$, where $y^\alpha := y_0^{\alpha_0} \dots y_n^{\alpha_n}$.

Lemma 2. $F^{(\lambda)} \in \mathcal{J}_{(0,0)}$, $\forall\ \lambda \in \mathbb{C}^* \implies F_k \in \mathcal{J}_{(0,0)}$, $\forall\ k \in \mathbb{N}$.

Proof. Let $A := (\mathcal{O}_{U \times \mathbb{C}^{n+1}})_{(0,0)}$. It is a Noetherian local ring. Set

$(y) := (y_0, \dots, y_n)$ and $J := \mathcal{J}_{(0,0)}$ as two ideals of A . For $\lambda \in \mathbb{C}^*$ let

$\mathrm{Jet}_m(F^{(\lambda)}) := \sum_{k=0}^m \lambda^k F_k$. Note that $F^{(\lambda)} - \mathrm{Jet}_m(F^{(\lambda)}) \in (y)^{m+1}$.

Fact 3. Krull's Theorem: $J = \bigcap_{m \geq m_0} (J + (y)^m)$, $\forall\ m_0 \geq 0$.

Since $\text{Jet}_m(F^{(\lambda)}) \in J + (y)^{m+1}$ for all $\lambda \in \mathbb{C}^*$, and by taking $m+1$

different values for $\lambda \Rightarrow F_k \in J + (y)^{m+1}$ for $k \leq m$. Fix $k \in \mathbb{N}$,

then $F_k \in \bigcap_{m \geq k+1} (J + (y)^m) = J$. ■

Therefore $\mathcal{J}_{(0,0)}$ is generated by elements of $A = \mathbb{C}\{x, y\}$ homogeneneous

in y . Since A is Noetherian, $\mathcal{J}_{(0,0)}$ is generated by a finite number of

these. They generate $\mathcal{J}$ over a nbhd of $(0,0)$ due to the coherency of

$\mathcal{J}$ and it remains to prove:

Lemma 3. If $\{F_j\} \subset \mathbb{C}\{x\}[y]$, are homogeneous in y and generate $\mathcal{J}$ over a nbhd of $(0,0)$, then they generate $\mathcal{I}$ over a nbhd of $\{0\} \times \mathbb{CP}^n$, i.e. that $\{F_j\}$ generate the stalk $\mathcal{I}_q$ for any $q \in \{0\} \times \mathbb{CP}^n$.

Proof. Say $q \in \{0\} \times \mathbb{CP}^n$ and $[\xi]$ are homogeneous coordinates on $\mathbb{CP}^n$ s.th. $q = (0, [1 : 0 : \ldots : 0])$; that the respective nonhomogeneous w and local (x, y_0, w) coordinates are on $W := \{\xi_0 \neq 0\}$ and on $\sigma_2^{-1}(U \times W)$ $\Rightarrow$ $\sigma_1(x, y_0, w) = (x, y_0, y_0 \cdot w)$, $\sigma_2(x, y_0, w) = (x, w)$. Say $G \in \mathcal{I}_q$ $\Rightarrow$ $\sigma_2^* G$ is a section of $\tilde{\mathcal{I}} = \sigma_2^* \mathcal{I}$ on a nbhd of $\sigma_2^{-1}(q) = \{(0, y_0, 0)\}_{y_0 \in \mathbb{C}}$.

$\{F_j\}$ generate $\mathcal{J}_{(0,0)}$ $\Rightarrow \{\sigma_1^* F_j\}$ generate $\tilde{\mathcal{J}}$ on a nbhd $V \subset U \times \tilde{\mathbb{C}}^{n+1}$ of $\sigma_1^{-1}(0,0) = \{0\} \times \mathbb{CP}^n$. Using $\sigma_1^{-1}(U \times \{0\}) = \{y_0 = 0\}$, Fact 2 and preceding it Note $\Rightarrow$ $\exists$ $d \in \mathbb{N}$ s.th. $y_0^d \sigma_2^* G \in \tilde{\mathcal{J}}_q$, i.e. $y_0^d \sigma_2^* G(x, y_0, w) = \sum_j A_j(x, y_0, w) \cdot \sigma_1^* F_j(x, y_0, w)$ with $\{A_j\} \subset \mathcal{H}$ on a nbhd of q . But $\sigma_2^* G(x, y_0, w) = G(x, w)$ and $\sigma_1^* F_j(x, y_0, w) = y_0^{d_j} F_j(x, 1, w)$ since the F_j are homogeneous in y of degrees d_j . Let $\mathbb{A}_j(x, w)$ be the coefficients at y_0^d in expansions of $y_0^{d_j} A_j(x, y_0, w)$. Then $\sum_j \mathbb{A}_j(x, w) F_j(x, 1, w) = G(x, w)$ $\Rightarrow$ $\{F_j(x, \xi)\}$ generate $\mathcal{I}$ on a nbhd of q , as required. ■