The de Rham Theorem

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January 8, 2010

The de Rham complex

Review.

Let M^n be a smooth, oriented, triangulated n-manifold.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d_{k-1}"] \arrow[d, "\mathrm{Int}^{k-1}"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{k}(M) \arrow[r, "d_k"] \arrow[d, "\mathrm{Int}^{k}"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{k+1}(M) \arrow[r] \arrow[d, "\mathrm{Int}^{k+1}"] & \cdots & \text{de Rham cochain complex} \\
\cdots \arrow[r] & \Sigma^*_{k-1} \arrow[r, "\partial^*_{k-1}"] & \Sigma^*_{k} \arrow[r, "\partial^*_{k}"] & \Sigma^*_{k+1} \arrow[r] & \cdots & \text{simplicial cochain complex}
\end{tikzcd}$$

$\mathrm{Int}^\bullet : \Omega^\bullet(M) \longrightarrow \Sigma^*_\bullet$ is a *morphism of cochain complexes*.

Theorem 1 (Elementary forms)

$\mathrm{Int}^\bullet$ admits a *right inverse*, i.e. $\exists\ \Phi^\bullet : \Sigma^*_\bullet \longrightarrow \Omega^\bullet(M)$ morphism of cochain complexes such that

$$\mathrm{Int}^k \circ \Phi^k = \mathrm{id}_{\Sigma^*_k} \quad \forall\, k.$$

The de Rham cohomology

Definition.

$\mathrm{H}^k(M)$	$:=$	$\ker d_k \,/\, \mathrm{im}\, d_{k-1}$	k^{th} *de Rham cohomology group*
$\mathrm{H}^k(\Sigma)$	$:=$	$\ker \partial_k^* \,/\, \mathrm{im}\, \partial_{k-1}^*$	k^{th} *cohomology group of* $\Sigma_\bullet$

Remark. As a morphism of cochain complexes, $\mathrm{Int}^k : \Omega^k(M) \longrightarrow \Sigma_\bullet^*$ induces a well-defined *homomorphism*

$$[\mathrm{Int}^k] : \mathrm{H}^k(M) \longrightarrow \mathrm{H}^k(\Sigma) \quad \forall\, k.$$

Remark. $\ker(\mathrm{Int}^\bullet)$ is a *subcomplex* of the de Rham cochain complex.

Lemma 1

The subcomplex $\ker(\mathrm{Int}^\bullet)$ is *acyclic*, i.e. $\mathrm{H}^k(\ker(\mathrm{Int}^\bullet)) = 0 \quad \forall\, k$.

Claim. *Lemma 1* is equivalent to

Lemma 1*

Let $\omega \in \Omega^k(M)$ be closed and $A \in \Sigma_{k-1}^*$ such that $\mathrm{Int}^k\, \omega = \partial_{k-1}^* A$.
Then $\exists\, \alpha \in \Omega^{k-1}(M)$ such that $d_{k-1}\alpha = \omega$ and $\mathrm{Int}^{k-1}\, \alpha = A$.

Proof of Claim. Assume that *Lemma 1* holds and that $d_k\omega = 0$, $\mathrm{Int}^k\,\omega = \partial^*_{k-1}A$. Then

$$\mathrm{Int}^k(d_{k-1}(\Phi^{k-1}A)) = \partial^*_{k-1}(\mathrm{Int}^{k-1}(\Phi^{k-1}A)) = \partial^*_{k-1}A = \mathrm{Int}^k\,\omega.$$

Setting

$$\beta := d_{k-1}(\Phi^{k-1}A) - \omega \in \ker(\mathrm{Int}^k) \cap \ker d_k,$$

we obtain,

$$[\beta] \in \mathrm{H}^k(\ker(\mathrm{Int}^\bullet)).$$

Lemma 1 impies that $[\beta] = 0$ and therefore, $\exists\,\gamma \in \ker(\mathrm{Int}^{k-1})$ with $d_{k-1}\gamma = \beta$.
Hence,

$$\omega = d_{k-1}(\Phi^{k-1}A - \gamma)$$

and, by setting $\alpha = \Phi^{k-1}A - \gamma$, we obtain

$$\mathrm{Int}^{k-1}\,\alpha = \mathrm{Int}^{k-1}(\Phi^{k-1}A) - \mathrm{Int}^{k-1}\,\gamma = A.$$

Assume now that *Lemma 1** is true and let $[\omega] \in \mathrm{H}^k(\ker(\mathrm{Int}^\bullet))$, i.e. $\omega \in \ker(\mathrm{Int}^k) \cap \ker d_k$. Then by *Lemma 1**, $\exists\,\alpha$ such that $d_{k-1}\alpha = \omega$ and $\alpha \in \ker(\mathrm{Int}^{k-1})$. It follows that $[\omega] = 0$. $\square$

Theorem 2 (de Rham)

$$[\mathrm{Int}^k] : \mathrm{H}^k(M) \longrightarrow \mathrm{H}^k(\Sigma) \quad \text{is an } \textit{isomorphism} \ \forall\, k.$$

Proof.

i) $[\mathrm{Int}^k]$ is *surjective*:
Let $[A] \in \mathrm{H}^k(\Sigma)$. Set $\omega := \Phi^k A \in \Omega^k(M)$. Since $d_k\omega = \Phi^{k+1}\partial_k^* A = 0$, $[\omega] \in \mathrm{H}^k(M)$.
Also, $[\mathrm{Int}^k][\omega] = [\mathrm{Int}^k \omega] = [\mathrm{Int}^k \Phi^k A] = [A]$.

ii) $[\mathrm{Int}^k]$ is *injective*:
Let $[\omega] \in \ker([\mathrm{Int}^k])$. Then $d_k\omega = 0$ and $[\mathrm{Int}^k \omega] = [\mathrm{Int}^k][\omega] = 0$, i.e. $\mathrm{Int}^k \omega \in \mathrm{im}\, \partial_{k-1}^*$.
*Lemma 1** implies that ω is exact and thus, $[\omega] = 0$. $\square$

Thus, for completing the proof of *de Rham*'s theorem, it remains to show that $\ker(\mathrm{Int}^\bullet)$ is acyclic.

For that, we need the following two lemmas:

Lemma 2 (Closed forms in star-shaped sets)

Let S be an open and *star-shaped* set in $\mathbb{R}^n$ and let ω be a closed k-form in S, $k > 0$. Then ω is exact.

Proof. Follows immediately from *Poincar's Lemma*, which we proved last time.

Lemma 3 (Extension of forms)

(a_k) Let ω be a closed k-form near $\partial\sigma$, where $\sigma = \sigma^s$ is an s-simplex in $\mathbb{R}^n$, $k \geq 0$, $s \geq 1$. Suppose that

$$\int_{\partial\sigma} \omega = 0 \qquad \text{if } s = k + 1. \tag{1}$$

Then there is a closed k-form $\tilde{\omega}$ near σ which extends ω.

(b_k) Let ω be a closed k-form near the s-simplex $\sigma = \sigma^s \subset \mathbb{R}^n$, $k \geq 1$, $s \geq 1$, and let α be a $(k-1)$-form near $\partial\sigma$ such that $d\alpha = \omega$ near $\partial\sigma$. Suppose that

$$\int_{\partial\sigma} \alpha = \int_{\sigma} \omega \qquad \text{if } s = k. \tag{2}$$

Then there is a $(k-1)$-form $\tilde{\alpha}$ near σ such that $\tilde{\alpha}$ extends α and $d\tilde{\alpha} = \omega$ near σ.

Proof by induction on k. We will show

i) (a_0) holds

ii) (a_{k-1}) $\Rightarrow$ (b_k)

iii) (b_k) $\Rightarrow$ (a_k), $k > 0$

i) Being a closed 0-form, ω is constant near any connected part of $\partial\sigma$.
If $s > 1$, then $\partial\sigma^s$ is connected and ω equals a constant c near $\partial\sigma$. Set $\tilde{\omega} = c$ near σ.
If $s = 1$, and say $\sigma^1 = p_0 p_1$, then

$$\omega(p_1) - \omega(p_0) = \int_{\partial\sigma} \omega = 0,$$

by (1), and thus, ω equals a constant c near $\partial\sigma$. Set $\tilde{\omega} = c$ near σ.

ii) Assume (a_{k-1}) holds and let ω, α be as in (b_k).
By choosing a star-shaped neighborhood of σ and applying *Lemma 2*, there exists a $(k-1)$-form α' near σ such that $d\alpha' = \omega$ near σ.
Set $\beta = \alpha - \alpha'$ near $\partial\sigma$ and observe that $d\beta = \omega - \omega = 0$.

Notice that, if $s = k$, (2) and *Stokes' Theorem* imply

$$\int_{\partial\sigma} \beta = \int_{\partial\sigma} \alpha - \int_{\partial\sigma} \alpha' = \int_{\sigma} \omega - \int_{\sigma} d\alpha' = 0.$$

By applying (a_{k-1}), we can extend β to $\tilde{\beta}$, which is defined near σ and closed. Setting $\tilde{\alpha} = \alpha' + \tilde{\beta}$ near σ, we obtain that $\tilde{\alpha}$ extends α and $d\tilde{\alpha} = \omega$ near σ as we wished.

iii) Assume (b_k), $k > 0$, holds and let ω be as in (a_k).
Say $\sigma = p_0 \dots p_s$ and set $\sigma' = p_1 \dots p_s$. Let $\mathcal{P}$ be the union of all proper faces of σ with p_0 as a vertex.
Choose now $\epsilon > 0$ small enough such that ω is defined in the ϵ-neighborhood $U_\epsilon(\mathcal{P})$ of $\mathcal{P}$.
Since $U_\epsilon(\mathcal{P})$ is star-shaped, by *Lemma 2*, there exists a $(k-1)$-form α' in $U_\epsilon(\mathcal{P})$ such that $d\alpha' = \omega$ in $U_\epsilon(\mathcal{P})$.
We have, in particular, $d\alpha' = \omega$ near $\partial\sigma'$.

If $s = k+1$, setting $A = \partial\sigma - \sigma'$, we obtain $\partial A = -\partial\sigma'$ and

$$\int_{\sigma'} \omega - \int_{\partial\sigma'} \alpha' = \int_{\sigma'} \omega + \int_{\partial A} \alpha' = \int_{\sigma'} \omega + \int_A d\alpha' = \int_{\partial\sigma} \omega = 0$$

by (1). We can now apply (b_k) and extend α' to $\tilde{\alpha}'$ near σ' such that $d\tilde{\alpha}' = \omega$ near σ'. It follows that there is a neighborhood $\widetilde{U}$ of $\partial\sigma'$ in which α' and $\tilde{\alpha}'$ are defined and equal. Set

$$\alpha = \begin{cases} \alpha'|_{\widetilde{U}} = \tilde{\alpha}'|_{\widetilde{U}} & \text{in } \widetilde{U} \\ \alpha' & \text{near } \mathcal{P}\backslash\widetilde{U} \\ \tilde{\alpha}' & \text{near } \sigma'\backslash\widetilde{U} \end{cases}$$

Observe that $d\alpha = \omega$ near $\partial\sigma$. By means of a partition of unity extend α to $\tilde{\alpha}$ near σ. $\tilde{\omega} := d\tilde{\alpha}$ satisfies the required properties. □

Definition. Let L^s denote the s-dimensional part of the triangulation of M, that is $L^s = \bigcup_i \sigma_i^s$.

Proof of Lemma 1.* We will define $\alpha_0, \ldots, \alpha_n$ such that

(a) α_s is defined near L^s, $s = 0, 1, \ldots, n$,

(b) $d\alpha_s = \omega$ near L^s, and $\alpha_s = \alpha_{s-1}$ near L^{s-1}, $s > 0$, and

(c) $\operatorname{Int} \alpha_{k-1} = A$.

Then, $\alpha := \alpha_n$ is the required form.

Construct α_s, $s = 0, 1, \ldots, n$, by induction on s:

By *Lemma 2*, there exists an α_0' near each vertex q_i such that $d\alpha_0' = \omega$.
If $k > 1$, set $\alpha_0 = \alpha_0'$.
If $k = 1$, for each vertex q_i choose a number b_i such that, setting $\alpha_0 = \alpha_0' + b_i$ near q_i,
$\operatorname{Int} \alpha_0 = A$.

Now suppose α_{s-1} has been constructed.

We will define α_s near each s-simplex such that (a), (b) and (c) hold. Since α_s is then fixed near L^{s-1}, we obtain a well-defined α_s near L^s.

Let σ be an s-simplex. Then $d\alpha_{s-1} = \omega$ near $\partial\sigma$, by construction.
If $s = k$, by (c),

$$\int_{\partial\sigma} \alpha_{k-1} = \operatorname{Int} \alpha_{k-1} \cdot \partial\sigma = A \cdot \partial\sigma = \partial^* A \cdot \sigma = \operatorname{Int} \omega \cdot \sigma = \int_\sigma \omega.$$

Since we can assume that M is embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$, we can now apply *Lemma 3*. It gives us a $(k-1)$-form $\tilde{\alpha}_s$ near σ such that $\tilde{\alpha}_s = \alpha_{s-1}$ near $\partial\sigma$ and $d\tilde{\alpha}_s = \omega$ near σ.
So (b) holds for $\tilde{\alpha}_s$.

If $s \neq k-1$, set $\alpha_s = \tilde{\alpha}_s$ near σ.
If $s = k-1$, define $B = A - \operatorname{Int} \tilde{\alpha}_{k-1}$ and set

$$\alpha_{k-1} = \tilde{\alpha}_{k-1} + \Phi B \quad \text{near } L^{k-1}.$$

To see that α_{k-1} satisfies (b), recall $Supp(\Phi \rho^*) \subset St(\rho)$ for each simplex ρ.
It follows that $\alpha_{k-1} = \tilde{\alpha}_{k-1}$ near L^{k-2} and thus, $\alpha_{k-1} = \alpha_{k-2}$ near L^{k-2}.
Also,

$$d\alpha_{k-1} = d\tilde{\alpha}_{k-1} + d\Phi B = d\tilde{\alpha}_{k-1} + \Phi \partial^* B = \omega \quad \text{near } L^{k-1}.$$

Since

$$\operatorname{Int} \alpha_{k-1} = \operatorname{Int} \tilde{\alpha}_{k-1} + B = A,$$

(c) holds and $\alpha_s = \alpha_{k-1}$ is as we wished. $\square$

An example: The Euler characteristic

Definition. Let M^n be a manifold. The *Euler characteristic* χ of M is the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k dim_{\mathbb{R}} H^k(\Omega).$$

The *de Rham Theorem* tells us that, no matter which triangulation we pick, the Euler characteristic equals the following:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k dim_{\mathbb{R}} H^k(\Sigma),$$

where

$$0 \longrightarrow \Sigma_0^* \xrightarrow{\partial_0^*} \Sigma_1^* \xrightarrow{\partial_1^*} \dots \xrightarrow{\partial_{n-2}^*} \Sigma_{n-1}^* \xrightarrow{\partial_{n-1}^*} \Sigma_n^* \longrightarrow 0$$

is the simplicial cochain complex according to the chosen triangulation of M^n.
Using

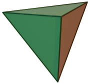
$$dim_{\mathbb{R}} H^k(\Sigma) = dim_{\mathbb{R}} \ker \partial_k^* - dim_{\mathbb{R}} \operatorname{im} \partial_{k-1}^* \quad \text{and} \quad dim_{\mathbb{R}} \Sigma_k^* = dim_{\mathbb{R}} \ker \partial_k^* + dim_{\mathbb{R}} \operatorname{im} \partial_k^*,$$

we finally obtain

$$\chi(M) = \sum_{k=0}^{n} (-1)^k dim_{\mathbb{R}} \Sigma_k^*,$$

that is simply the alternating sum of the number of the k-dimensional faces, $k = 0, 1, \cdots, n$.

Example 1. 2-**Sphere**

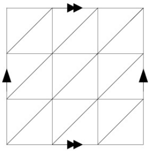


We can use a tetrahedron T to triangulate S^2.
Then

$$\begin{aligned}\chi(S^2) &= \text{number of vertices of } T - \text{number of edges of } T + \text{number of faces of } T \\ &= 4 - 6 - 4 \quad = \quad 2.\end{aligned}$$

Example 2. 2-**Torus**

Triangulate the torus in the following way:



K

$$\begin{aligned}\chi(\mathbb{T}^2) &= \text{number of vertices of } K - \text{number of edges of } K + \text{number of faces of } K \\ &= 9 - 27 + 18 \quad = \quad 0\end{aligned}$$