MATH 1045HF INTRODUCTION TO ERGODIC THEORY

GIULIO TIOZZO

This class provides an introduction to classical topics in ergodic theory, with applications to dynamical systems.

Overview. Broadly speaking, ergodic theory is the study of measure preserving transformations. In many situations in dynamical systems, it is impossible to describe the behaviour of *all* trajectories, so it is more profitable to look at the behaviour of *most* trajectories, or *typical* trajectories. Probability helps us make sense of what this means in a rigorous way, and this leads to the study of *measure preserving transformations*, which is the subject of ergodic theory.

We will start with a basic introduction to the terminology and setup of ergodic theory, and we will prove the classical *ergodic theorems*, i.e. theorems which assure us that averages exist: in particular, we will prove the ergodic theorems of Birkhoff, Von Neumann, and Kingman. Then we will study the different notions of ergodicity and the related notion of *mixing*.

In the second part, we will focus on the definition of *entropy*, which is a fundamental quantity to measure how chaotic a dynamical system is: we will see the measure-theoretic notion (Kolmogorov-Sinai entropy) as well as the topological version, and see that they are connected by the *variational principle*.

In the third part, we will see how to use ergodic theory to *count* things, for instance closed geodesics on a manifold of negative curvature. In order to do this, we will prove that the geodesic flow is ergodic, and we will introduce *dynamical* ζ *functions* and the *Ruelle transfer operator*. By studying the spectral theory for such an operator we will prove the *prime orbit theorem*, i.e. the analog of the prime number theorem for closed geodesics on a hyperbolic manifold.

Prerequisites. An introduction to measure theory and/or probability and basic topology.

Location. Mon 12-1 in BA6180 and Fri 12-2 in BA6183.

Instructor. Giulio Tiozzo, `tiozzo@math.utoronto.ca`. Some additional information may be posted on my website `http://www.math.toronto.edu/tiozzo/` especially under *Teaching*.

Office hours. Mondays, 4.30-5.30 PM, in BA6206.

Grading. If you need a grade for this class, talk to me and we will arrange for you to give some presentations towards the end of the course.

Lecture plan.

(1) Sep 10 - Measure preserving transformations
(2) Sep 14 - Poincaré recurrence and induced transformations
(3) Sep 17 - Ergodicity
(4) Sep 21 - The ergodic theorem
(5) Sep 24 - Irreducible Markov chains
(6) Sep 28 - Mixing

(7) Oct 1 - Measure preserving isomorphisms and factor maps
(8) Oct 5 - Entropy - definition and first properties
(9) Oct 8 - no class (Thanksgiving)
(10) Oct 12 - The Shannon-Breiman-McMillan theorem
(11) Oct 15 - The Hurewicz ergodic theorem
(12) Oct 19 - Invariant measures
(13) Oct 22 - Unique ergodicity
(14) Oct 26 - Topological dynamics
(15) Oct 29 - Topological entropy
(16) Nov 2 - The variational principle
(17) Nov 5 - Measures of maximal entropy
(18) Nov 5 - Ergodicity of geodesic flows
(19) Nov 9 - The Perron-Frobenius theorem
(20) Nov 12 - no class
(21) Nov 16 - no class
(22) Nov 19 - The Ruelle operator
(23) Nov 23 - Gibbs measures and pressure
(24) Nov 26 - The complex Ruelle operator
(25) Nov 30 - Periodic points and zeta functions
(26) Dec 3 - The prime orbit theorem

Bibliography.
The main reference for the first part of the course is

- K. Dajani, S. Dirskin, *A simple introduction to ergodic theory*, available at `http://www.staff.science.uu.nl/~kraai101/lecturenotes2009.pdf`

while for the second part (on thermodynamic formalism) see

- W. Parry, M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187-88** (1990), available at `http://homepages.warwick.ac.uk/~masdbl/PP.pdf`

As a general reference, see

- A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.
- P. Walters, *An introduction to Ergodic Theory*, Springer, 1982.

while here are some papers on the ergodic theory of geodesic flows:

- W. Ballmann, M. Brin, *On the ergodicity of geodesic flows*, Ergodic Th. Dynam. Sys. **2** (1982), 311-315.
- E. Hopf, *Ergodic theory and the geodesic flow on surfaces of negative curvature*, Bull. A.M.S. **6** (1971), no. 6, 863-877.
- A. Katok. *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.E.S. **51** (1980), 137-173.
- V. I. Oseledec, *A multiplicative ergodic theorem*, Trans. Moscow Math. Soc. **19** (1968).
- Ya. B. Pesin, *Geodesic flows on closed Riemannian manifolds without focal points*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 6, 1252-1288.