Plateaux and tuning for the entropy of α-continued fraction transformations

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In a nutshell

We study the bifurcation locus $\mathcal{E}$ for the one-parameter family of α-continued fraction transformations T_α, and describe its self-similar structure in terms of *tuning operators*.
This combinatorial description is used to characterize the plateaux of the entropy function $h(\alpha)$ and to explain the fractal structure observed in the graph of the entropy.

Introduction

The family $\{T_\alpha\}_{\alpha\in(0,1]}$ of α-continued fraction transformations is a family of discontinuous interval maps, which generalize the well-known Gauss map.

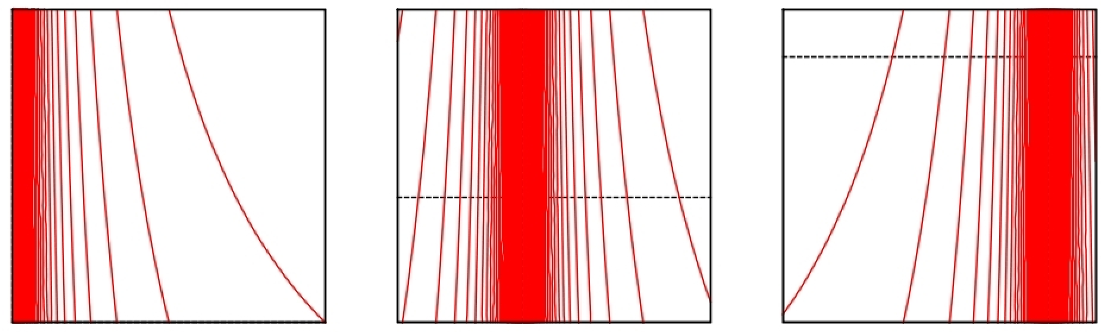

Figure 1: Graphs of T_α for $\alpha = 1, \alpha = 0.6, \alpha = 0.15$.

$T_\alpha : [\alpha-1, \alpha] \to [\alpha-1, \alpha]$ is defined as

$$T_\alpha(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$$

Entropy

For all $\alpha > 0$ the map T_α admits an invariant measure μ_α absolutely continuous w.r.t. Lebesgue.
We shall study the entropy $h(\alpha)$ of T_α with respect to μ_α; it is a continuous function, piecewise smooth for $\alpha \geq 1/2$.

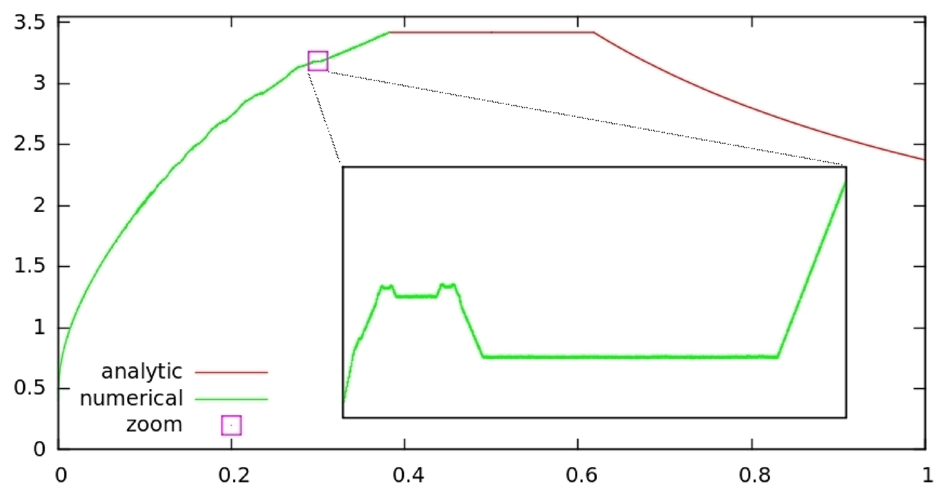

Figure 2: $h(\alpha)$ as a function of α. You can see a phase transition for $\alpha = \frac{\sqrt{5}-1}{2}$, and infinitely many others as you zoom in.

For $0 < \alpha < 1/2$ the entropy has a more complex behaviour; Nakada and Natsui [NN] proved that the entropy is not monotone on any interval $[0, c]$.

Thickening $\mathbb{Q}$

Every $r \in (0,1) \cap \mathbb{Q}$ admits exactly two continued fraction expansions; in this way, one can associate to every rational value, two finite strings S_0 and S_1 of positive integers, S_0 has even length and S_1 odd length. For instance, since $3/10 = [0;3,3] = [0;3,2,1]$, the two strings associated to $3/10$ will be $S_0 = (3,3)$ and $S_1 = (3,2,1)$.
Now, for each $r \in \mathbb{Q} \cap (0,1)$ we define the *quadratic interval* associated to r as the open interval

$$I_r := (\alpha_1, \alpha_0)$$

whose endpoints are the two quadratic irrationals $\alpha_0 = [0; \overline{S_0}]$ and $\alpha_1 = [0; \overline{S_1}]$. Let us set

$$\mathcal{E} := [0,1] \setminus \bigcup_{r\in(0,1]\cap\mathbb{Q}} I_r$$

The largest element of $\mathcal{E}$ is $g := (\sqrt{5}-1)/2$, which corresponds to the rightmost phase transition of the entropy (Figure 2).

The connected components of $[0,1] \setminus \mathcal{E}$ are themselves quadratic intervals, called *maximal* quadratic intervals; the set $\mathbb{Q}_E$ of *extremal rational numbers* is

$$\mathbb{Q}_E := \{r \in (0,1] \ : \ I_r \text{ is maximal}\}$$

Matching and monotonicity

For all parameters belonging to a maximal quadratic interval, the transformation T_α satisfies a *matching condition* which implies monotonicity of the entropy:

Theorem ([CT]). *Let I_r be a maximal quadratic interval, and $r = [0; a_1, \dots, a_n]$ with n even. Let*

$$N = \sum_{i \text{ even}} a_i \qquad M = \sum_{i \text{ odd}} a_i \qquad [\![r]\!] := M - N$$

Then

$$T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1) \qquad \forall \alpha \in I_r$$

The sign of the matching index $[\![r]\!]$ determines the monotonicity of entropy on I_r:
- if $[\![r]\!] > 0$, the entropy $h(\alpha)$ is increasing on I_r,
- if $[\![r]\!] = 0$ it is constant,
- if $[\![r]\!] < 0$ it is decreasing.

Dictionary

There is an explicit correspondence between maximal quadratic intervals and real hyperbolic components of the Mandelbrot set $\mathcal{M}$: in other words, the bifurcation set $\mathcal{E}$ has the same combinatorial structure as the real section of the boundary of $\mathcal{M}$.

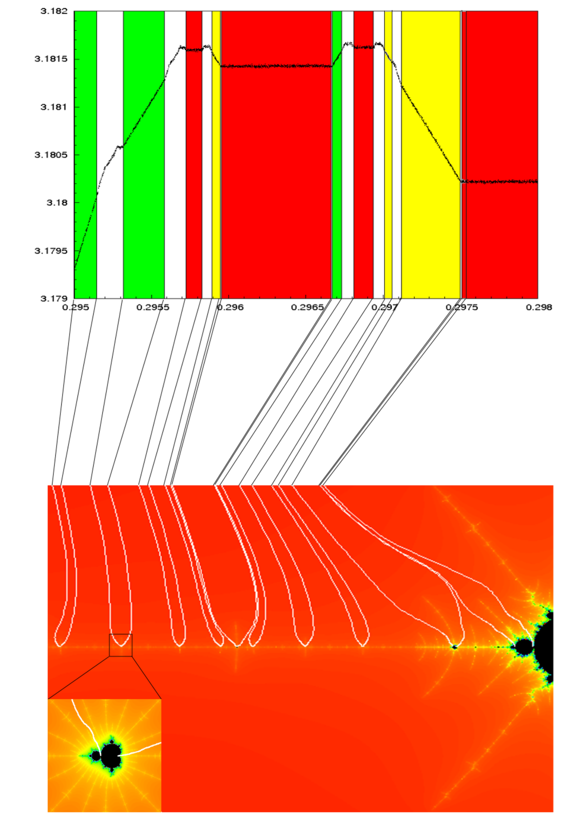

Figure 3: On the top, the entropy of α-c.f. as a function of α; colored strips correspond to maximal intervals. At the bottom, a section of the Mandelbrot set along the real line, with external rays landing on the real axis.

Tuning

In the Mandelbrot set, the baby copies of $\mathcal{M}$ are homeomorphic images of the whole $\mathcal{M}$ via the Douady-Hubbard tuning operators.
Analogously, we define tuning operators acting on the parameter space of α-continued fraction transformations.
For any $r \in \mathbb{Q}_E$ let us define the *tuning window* $W_r := [\omega, \alpha_0]$ with $\omega := [0; S_1\overline{S_0}]$, $\alpha_0 := [0;\overline{S_0}]$ and the tuning operator

$$\tau_r([0; a_1, a_2, \dots]) = [0; S_1 S_0^{a_1 - 1} S_1 S_0^{a_2-1} \dots]$$

This map is order-preserving and it maps $\mathcal{E}$ inside itself. Since $\tau_r(\mathcal{E}) = \mathcal{E} \cap W_r$, τ_r induces a bijection between all maximal intervals and those contained in W_r; if I_p is a maximal interval then $\tau_r(I_p) = I_{\tau_r(p)}$ and

$$[\![\tau_r(p)]\!] = -[\![r]\!][\![p]\!].$$

A tuning window r is called *neutral* if $[\![r]\!] = 0$; if W_r is a *neutral tuning window* then the formula above shows that the entropy is locally constant on $W_r \setminus \mathcal{E}$.

Plateaux and monotonicity

Using tuning windows, we can explain the self-similar structure of h and characterize the plateaux:

Theorem. *Let W_r be a non-neutral tuning window. Then the monotonicity of the entropy on W_r reflects the monotonicity on the whole parameter space* $[0,1]$.

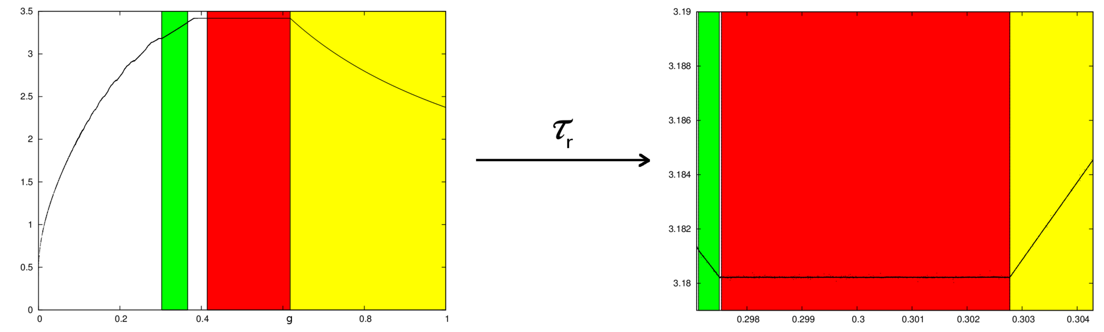

Figure 4: Illustration of the theorem in the case $r = 1/3$: on the left, three maximal intervals and their tuned images on the right: since $[\![r]\!] > 0$, the monotonicity on corresponding intervals is reversed (e.g., entropy is increasing on the green interval on the left and decreasing on its image on the right).

Theorem. *Let a* plateau *be a maximal connected set over which the entropy is constant. Then, every plateau of h is a neutral tuning window W_r.*

For instance, the plateau $[g^2, g]$ (see Figure 2) is the neutral tuning window $W_{1/2}$.

Questions

- Is $h(1/2)$ the global maximum of h?
- Is the entropy smooth outside $\mathcal{E}$?
- Can we use use this combinatorial technique to study the Mandelbrot set?
- Can we extend the dictionary to a correspondence between limit sets of Fuchsian semigroups and rays landing on Julia sets?

References

References

[CT] C Carminati, G Tiozzo, *A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fractions*, to appear in Ergodic Theory Dynam. Systems.

[NN] H Nakada, R Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity **21** (2008), 1207–1225.