Topological entropy of quadratic polynomials and sections of the Mandelbrot set

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Summary

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2. External rays

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3. Main theorem

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5. Complex version

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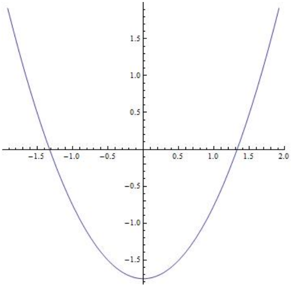

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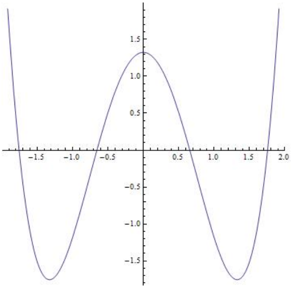

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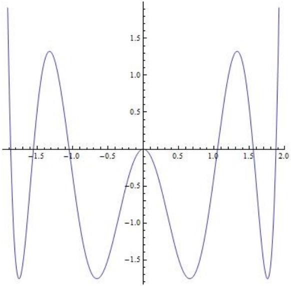

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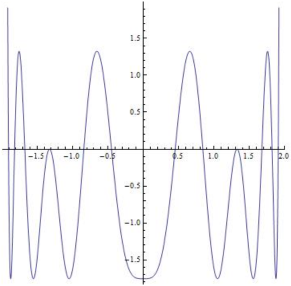

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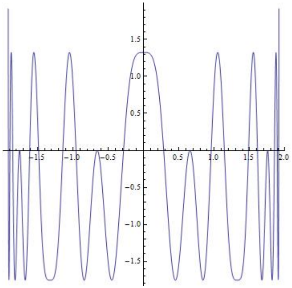

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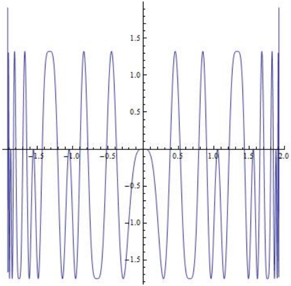

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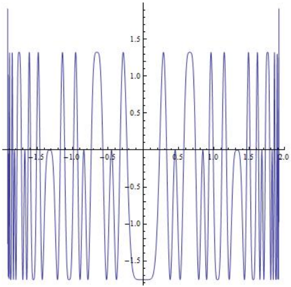

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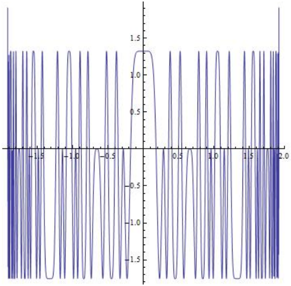

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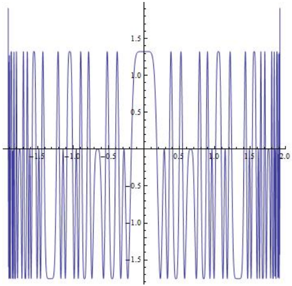

Entropy measures the randomness of the dynamics.

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How does entropy change with the parameter c?

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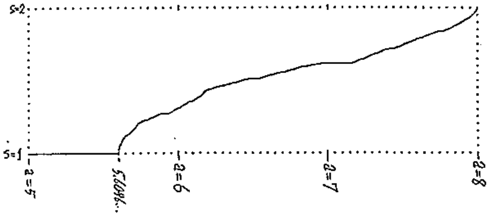

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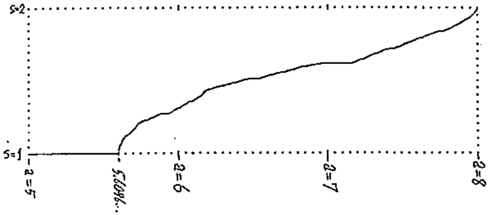
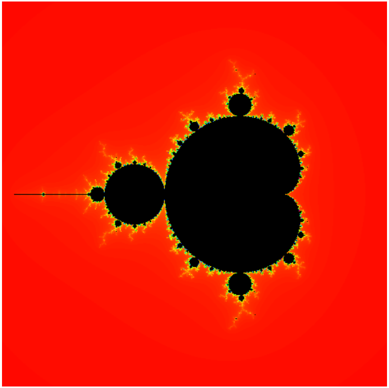


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$.

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \nrightarrow \infty\}$$



External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

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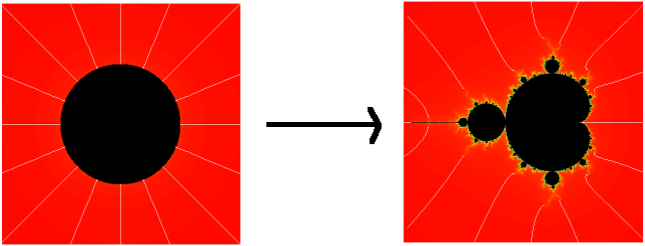

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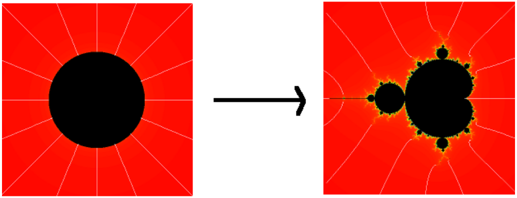

Harmonic measure

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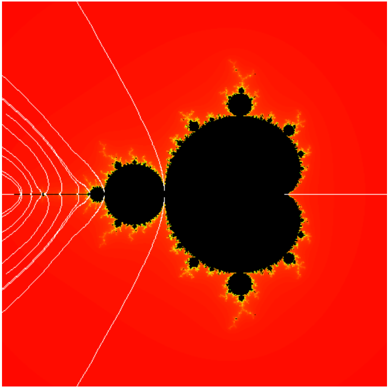
For instance, take $A = \mathcal{M} \cap \mathbb{R}$ the real section of the Mandelbrot set. How common is it for a ray to land on the real axis?



Real section of the Mandelbrot set

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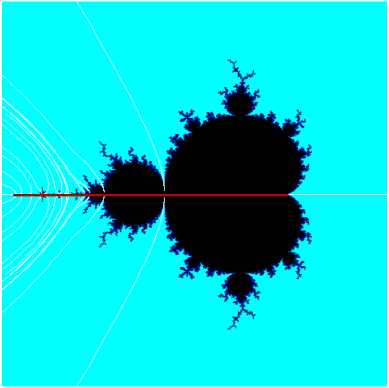

Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

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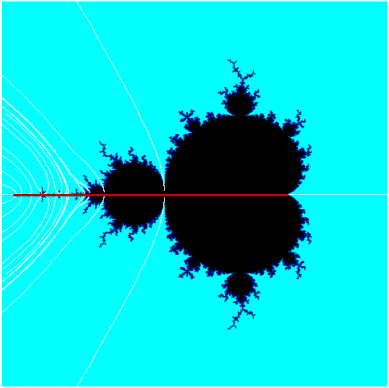

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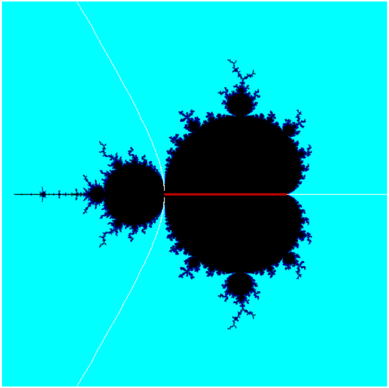

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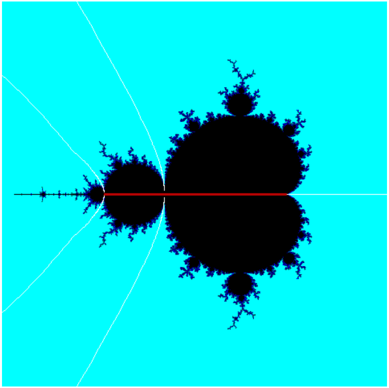

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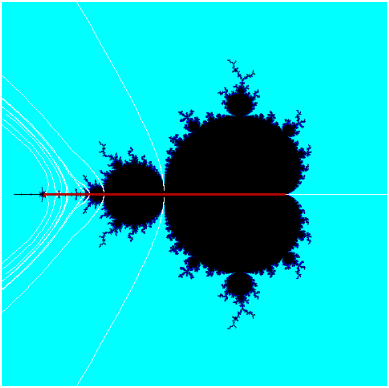

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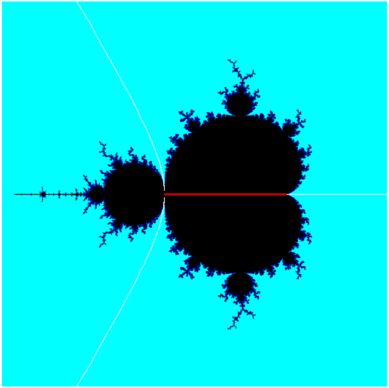
The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.



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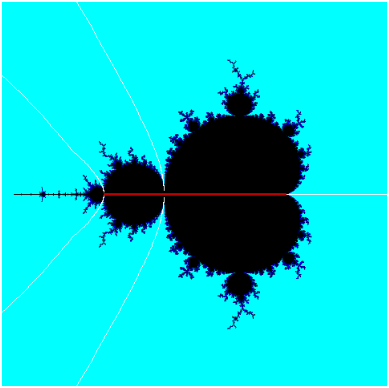

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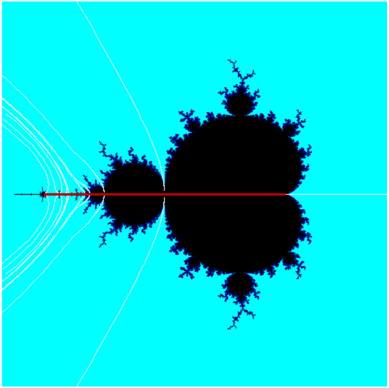

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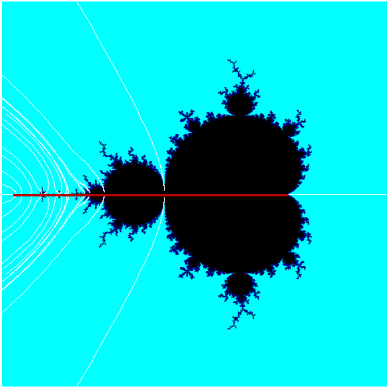

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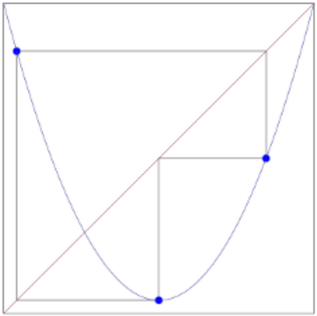

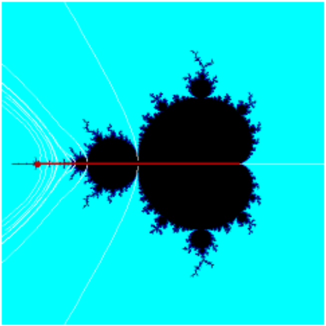

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- It does not depend on MLC.
- It can be generalized to (some) non-real veins.

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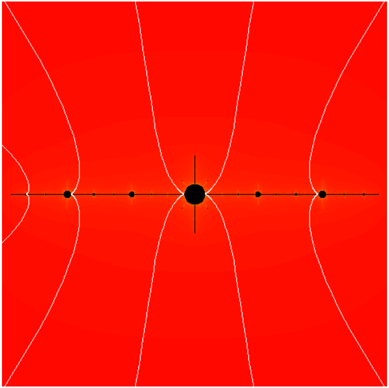

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Corollary

The set of biaccessible angles for the Feigenbaum parameter (limit of period doubling cascades) c_{Feig} has Hausdorff dimension 0.

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5. By renormalization, the same holds for all parameters which are not infinitely renormalizable.

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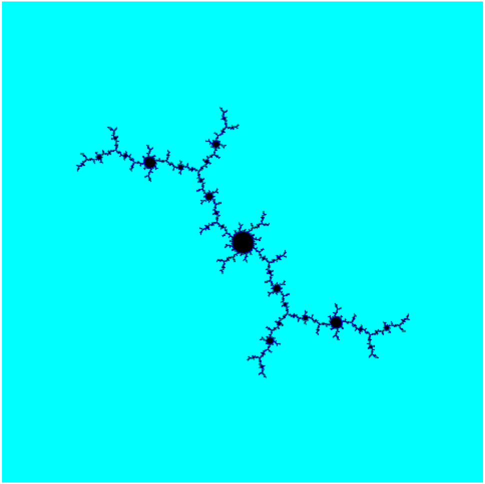
If c is a dominant parameter, then for each $c' > c$, P_c contains a diffeomorphic copy of $H_{c'}$.

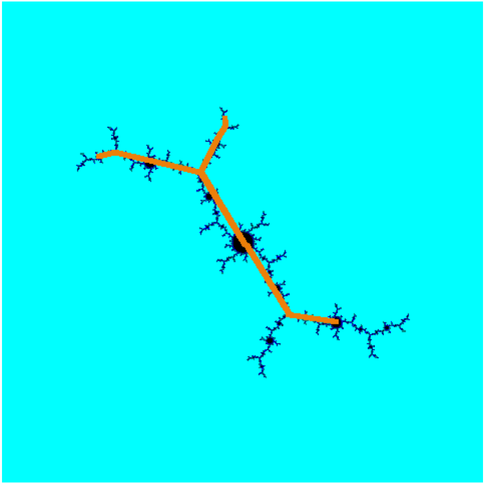
4. We prove dominant parameters are dense in the set of non-renormalizable parameters.
5. By renormalization, the same holds for all parameters which are not infinitely renormalizable.
6. By density of such parameters, the result holds.

The complex case: Hubbard trees



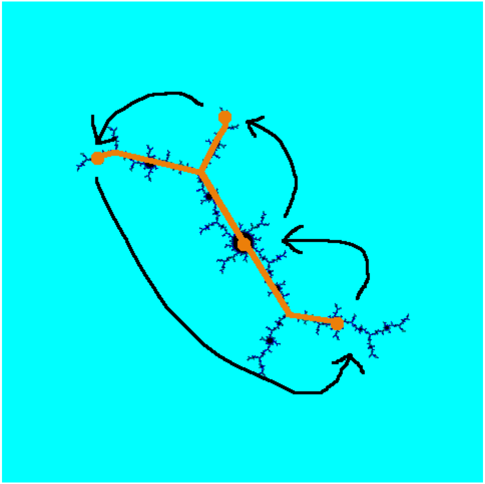
Complex Hubbard trees

The Hubbard tree H_c of a quadratic polynomial is a forward invariant subset of the full Julia set which contains the critical orbit.

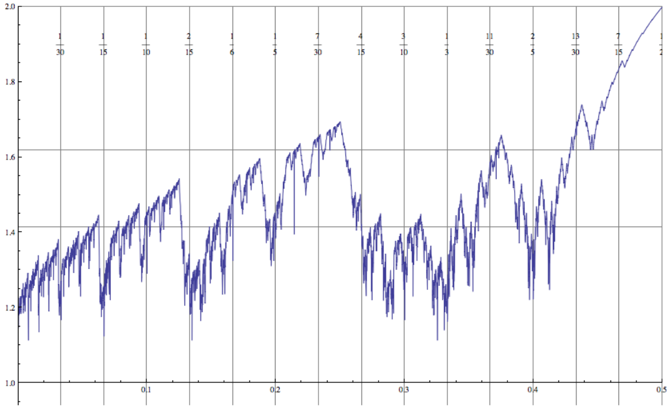


Complex Hubbard trees

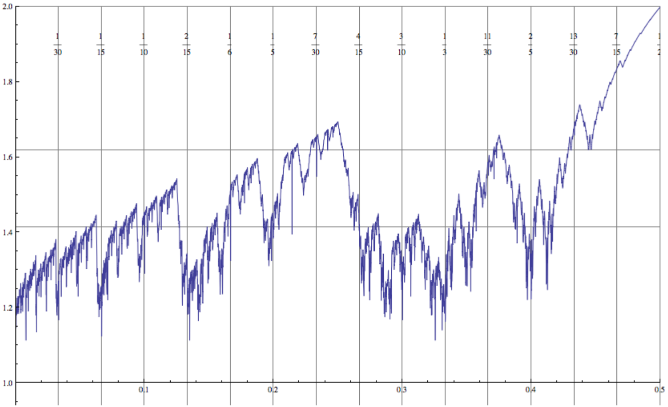
The Hubbard tree H_c of a quadratic polynomial is a forward invariant subset of the full Julia set which contains the critical orbit. The map f_c acts on it.



Entropy of Hubbard trees as a function of external angle



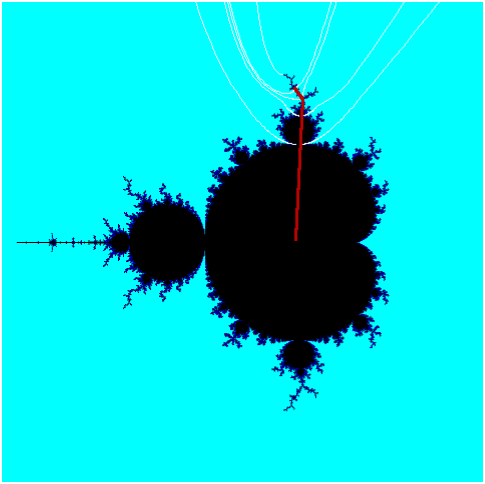
Entropy of Hubbard trees as a function of external angle



Can you see the Mandelbrot set in this picture?

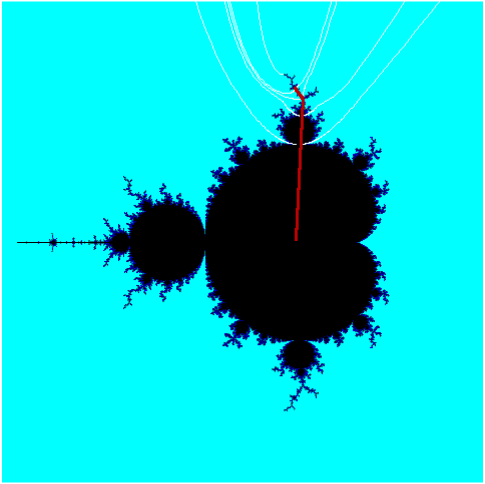
The complex case

A vein is an embedded arc in the Mandelbrot set.



The complex case

A vein is an embedded arc in the Mandelbrot set.

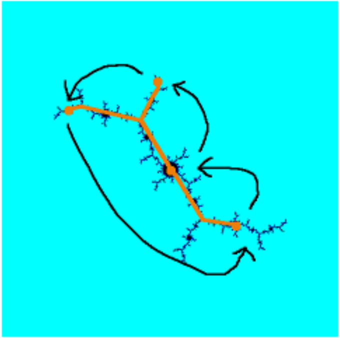


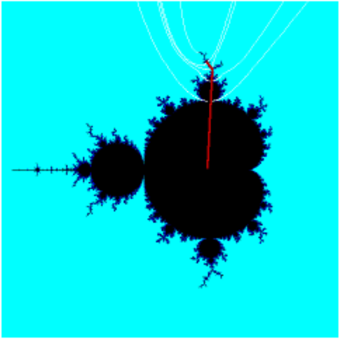
Given a parameter c along a vein, we can look at the set P_c of parameter rays which land on the vein below c.

Complex version

Let γ be the principal vein in the p/q-limb of the Mandelbrot set, and let $c \in \gamma$. Then

$$\frac{h_{top}(f_c, H_c)}{\log 2} = \text{H.dim } P_c$$





The end

Thank you!

A unified approach

The dictionary yields a unified proof of the following results:

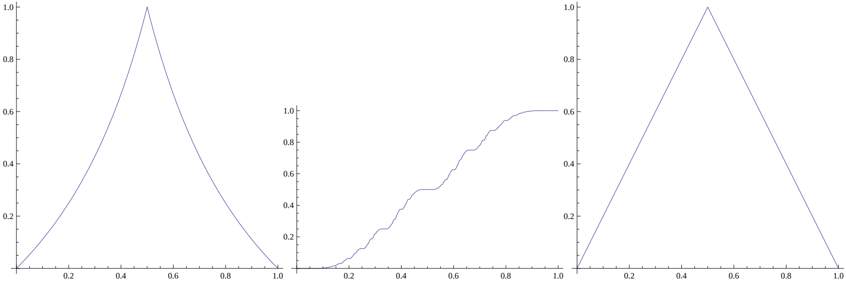
1. The set of matching intervals for α-continued fractions has zero measure and full Hausdorff dimension (Nakada-Natsui conjecture, CT 2010)
2. The real part of the boundary of the Mandelbrot set has Hausdorff dimension 1

$$H.dim(\partial\mathcal{M} \cap \mathbb{R}) = 1$$

   (Zakeri, 2000)
3. The set of univoque numbers has zero measure and full Hausdorff dimension (Erdős-Horváth-Joó, Daróczy-Kátai, Komornik-Loreti)

From Farey to the tent map, via ?

Minkowski's question-mark function conjugates the Farey map with the tent map



Continued fractions $\Leftrightarrow$ **Binary expansions**