Statistical properties for coarse expanding dynamical systems

Giulio Tiozzo
University of Toronto

Quasiworld workshop
July 14^{th}, 2020

Summary

1. Thurston maps, expanding or not

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism
5. Statement of results

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism
5. Statement of results
6. Question time

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism
5. Statement of results
6. Question time
7. Proof: the case without periodic critical points

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism
5. Statement of results
6. Question time
7. Proof: the case without periodic critical points
8. Proof: how to deal with periodic critical points

Summary

1. Thurston maps, expanding or not
2. Coarse expanding systems
3. Examples of expanding systems
4. Thermodynamic formalism
5. Statement of results
6. Question time
7. Proof: the case without periodic critical points
8. Proof: how to deal with periodic critical points

joint with T. Das, F. Przytycki, M. Urbański, A. Zdunik

Thurston maps

A Thurston map is a postcritically finite branched cover

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\#P(f) < +\infty$.

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\#P(f) < +\infty$.

Thurston's theorem: combinatorial characterization of Thurston maps which are "equivalent" to rational maps.

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\#P(f) < +\infty$.

Thurston's theorem: combinatorial characterization of Thurston maps which are "equivalent" to rational maps.

Examples

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\# P(f) < +\infty$.

Thurston's theorem: combinatorial characterization of Thurston maps which are "equivalent" to rational maps.

Examples

- $f(z) := \frac{p(z)}{q(z)}$ a postcritically finite rational map.

Thurston maps

A Thurston map is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\#P(f) < +\infty$.

Thurston's theorem: combinatorial characterization of Thurston maps which are "equivalent" to rational maps.

Examples

- $f(z) := \frac{p(z)}{q(z)}$ a postcritically finite rational map.
- A Lattés map

Thurston maps

A <u>Thurston map</u> is a postcritically finite branched cover $f : S^2 \to S^2$ of the 2-sphere.

Recall

$$P(f) := \bigcup_{n \geq 1} f^n(\text{Crit } f)$$

and f is postcritically finite if $\#P(f) < +\infty$.

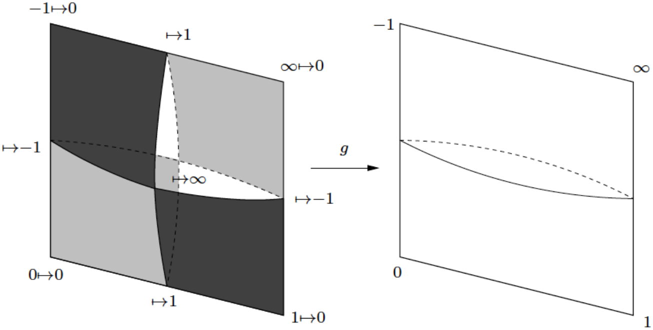
Thurston's theorem: combinatorial characterization of Thurston maps which are "equivalent" to rational maps.

<u>Examples</u>

- $f(z) := \frac{p(z)}{q(z)}$ a postcritically finite rational map.
- A Lattés map $g(z) = 2z$ as $g : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ and $z \sim -z$

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{g} & \mathbb{C}/\Lambda \\ \downarrow & & \downarrow \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

Lattés maps



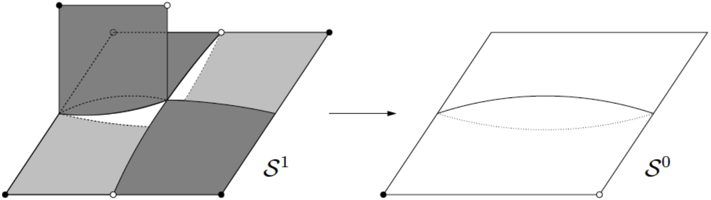
$$g(z) = 4\frac{z(1-z^2)}{(1+z^2)^2}$$

Pillow maps with flaps

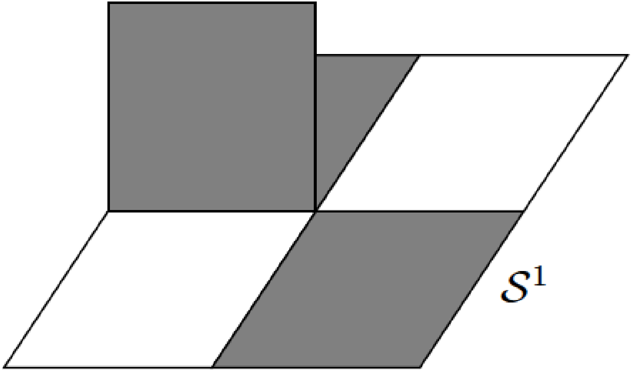
Modification of Lattés maps: add a “flap”

Pillow maps with flaps

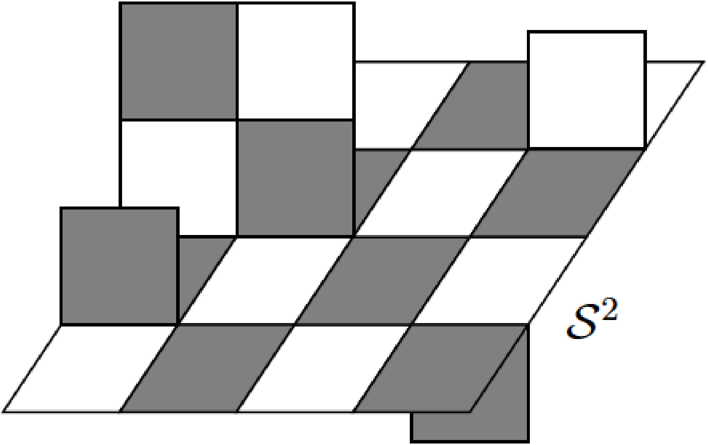
Modification of Lattés maps: add a “flap”



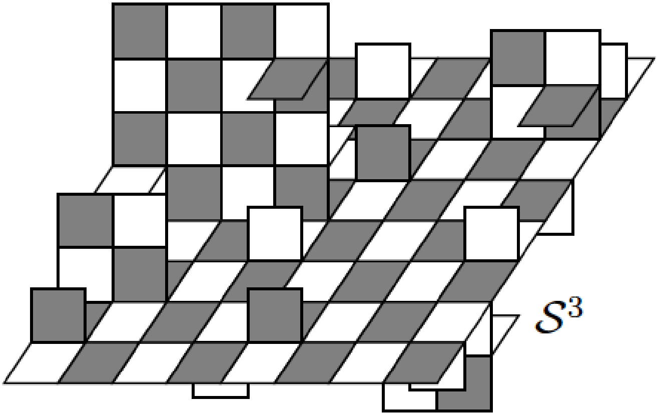
Pillows with flaps



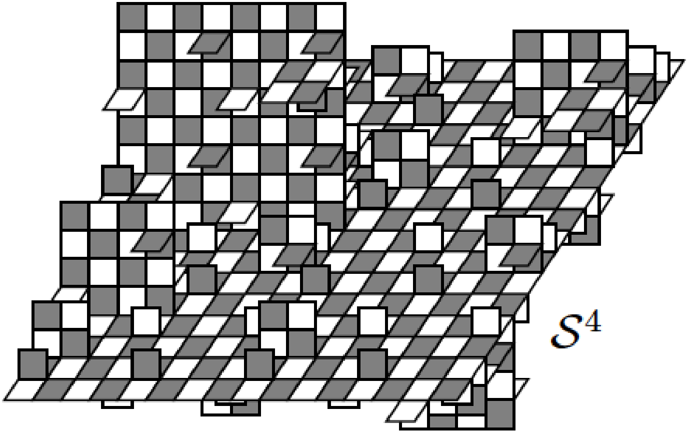
Pillows with flaps



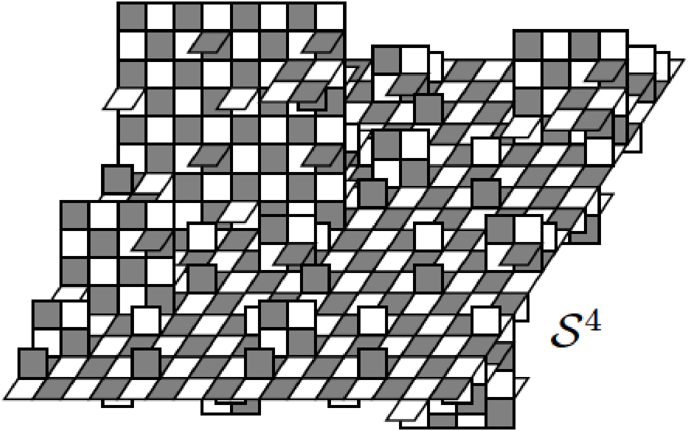
Pillows with flaps



Pillows with flaps

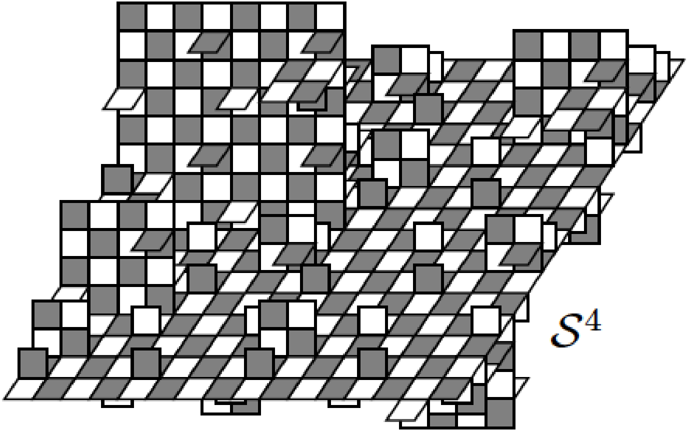


Pillows with flaps



Visual metric on the sphere:

Pillows with flaps



Visual metric on the sphere:

$$\text{diam(piece of depth } n) \cong \lambda^{-n}$$

History

- Bonk-Meyer: Expanding Thurston maps.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space,

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.
- Przytycki, Urbański: ergodic theory of rational maps for hyperbolic potentials

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.

- Przytycki, Urbański: ergodic theory of rational maps for hyperbolic potentials
- Rivera-Letelier, Inoquio-Renteria: ergodic theory of Collet-Eckmann rational maps

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.

- Przytycki, Urbański: ergodic theory of rational maps for hyperbolic potentials
- Rivera-Letelier, Inoquio-Renteria: ergodic theory of Collet-Eckmann rational maps
- Z. Li: ergodic theory of expanding Thurston maps.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.

- Przytycki, Urbański: ergodic theory of rational maps for hyperbolic potentials
- Rivera-Letelier, Inoquio-Renteria: ergodic theory of Collet-Eckmann rational maps
- Z. Li: ergodic theory of expanding Thurston maps. Existence and uniqueness of equilibrium measures.

History

- Bonk-Meyer: Expanding Thurston maps. Rigidity for measure of maximal entropy.
- Haïssinsky-Pilgrim: coarse expanding conformal (cxc) systems. General axioms in metric space, not postcritically finite. Existence and uniqueness of measure of maximal entropy.

- Przytycki, Urbański: ergodic theory of rational maps for hyperbolic potentials
- Rivera-Letelier, Inoquio-Renteria: ergodic theory of Collet-Eckmann rational maps
- Z. Li: ergodic theory of expanding Thurston maps. Existence and uniqueness of equilibrium measures. Note: he works directly on the sphere, estimating the PF operator.

Finite branched coverings

Let $f : Y \to Z$ be a continuous map of loc. connected spaces.

Finite branched coverings

Let $f : Y \to Z$ be a continuous map of loc. connected spaces. The degree of f is defined as

$$\deg(f) := \sup\{\# f^{-1}(z) \ : \ z \in Z\}.$$

Finite branched coverings

Let $f : Y \to Z$ be a continuous map of loc. connected spaces. The degree of f is defined as

$$\deg(f) := \sup\{\# f^{-1}(z) \ : \ z \in Z\}.$$

Given a point $y \in Y$, the local degree of f at y is

$$\deg(f;y) := \inf_U \sup \#\{f^{-1}(z) \cap U \ : \ z \in f(U)\}$$

Finite branched coverings

Let $f : Y \to Z$ be a continuous map of loc. connected spaces. The degree of f is defined as

$$\deg(f) := \sup\{\# f^{-1}(z) \ : \ z \in Z\}.$$

Given a point $y \in Y$, the local degree of f at y is

$$\deg(f; y) := \inf_U \sup \#\{f^{-1}(z) \cap U \ : \ z \in f(U)\}$$

where U ranges over all open neighborhoods of y.

Finite branched coverings

Let $f : Y \to Z$ be a continuous map of loc. connected spaces.
The degree of f is defined as

$$\deg(f) := \sup\{\# f^{-1}(z) \ : \ z \in Z\}.$$

Given a point $y \in Y$, the local degree of f at y is

$$\deg(f; y) := \inf_U \sup \#\{f^{-1}(z) \cap U \ : \ z \in f(U)\}$$

where U ranges over all open neighborhoods of y.
A point y is critical if $\deg(f; y) > 1$.

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d if $\deg(f) = d < \infty$ and

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d if $\deg(f) = d < \infty$ and

1. for any $z \in Z$,

$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d if $\deg(f) = d < \infty$ and

1. for any $z \in Z$,

$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$

2. for any $y_0 \in Y$ there are compact neighborhoods U, V of y_0 and $f(y_0)$

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d if $\deg(f) = d < \infty$ and

1. for any $z \in Z$,
$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$

2. for any $y_0 \in Y$ there are compact neighborhoods U, V of y_0 and $f(y_0)$ such that
$$\sum_{y \in U, f(y) = z} \deg(f; y) = \deg(f; y_0)$$
for all $z \in V$.

Finite branched coverings

Definition

The map $f : Y \to Z$ is a finite branched cover of degree d if $\deg(f) = d < \infty$ and

1. for any $z \in Z$,

$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$

2. for any $y_0 \in Y$ there are compact neighborhoods U, V of y_0 and $f(y_0)$ such that

$$\sum_{y \in U, f(y) = z} \deg(f; y) = \deg(f; y_0)$$

for all $z \in V$.

We define the branch set as $B_f := \{y \in Y \ : \ \deg(f; y) > 1\}$

The setting

Let $f : W_1 \to W_0$ be a finite branched covering.

The setting

Let $f : W_1 \to W_0$ be a finite branched covering.

Definition

The <u>repellor</u> is

The setting

Let $f : W_1 \to W_0$ be a finite branched covering.

Definition

The repellor is

$$X := \bigcap_{n \geq 0} f^{-n}(W_1)$$

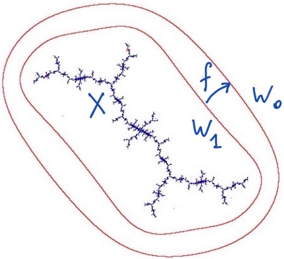
The setting

Let $f : W_1 \to W_0$ be a finite branched covering.

Definition

The repellor is

$$X := \bigcap_{n \geq 0} f^{-n}(W_1)$$



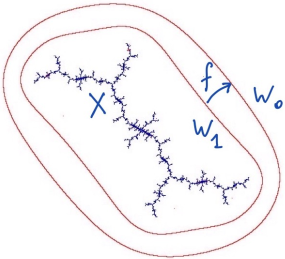
The setting

Let $f : W_1 \to W_0$ be a finite branched covering.

Definition

The repellor is

$$X := \bigcap_{n \geq 0} f^{-n}(W_1)$$



If $\mathcal{U}$ is a cover of W_1, then

$$\mathcal{U}_n := \{\text{connected components of } f^{-n}(\mathcal{U})\}$$

Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

$$\text{``diam}(\mathcal{U}_n) \to 0\text{''}$$

Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.

Coarse expanding conformal (cxc) systems

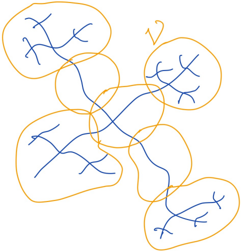
Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.



Coarse expanding conformal (cxc) systems

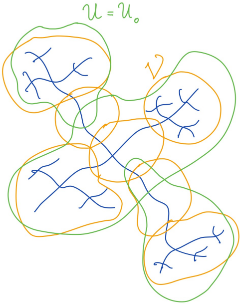
Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

$$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.



Coarse expanding conformal (cxc) systems

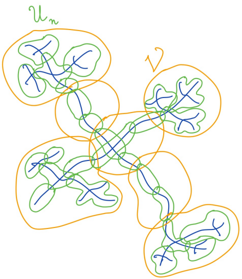
Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

$$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.



Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{"diam}(\mathcal{U}_n) \to 0\text{"}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.
- **[Irreducibility]**: For any $x \in X$ and any open set $W \ni x$, there exists n such that $f^n(W) \supseteq X$.

Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms
A finite branched cover $f : W_1 \to W_0$ which satisfies:

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.
- **[Irreducibility]**: For any $x \in X$ and any open set $W \ni x$, there exists n such that $f^n(W) \supseteq X$.
- **[Degree]**: $\exists C > 0$ s.t.

  $$\deg(f^k : U \to V) \leq C$$

  for any $U \in \mathcal{U}_{n+k}$, $V \in \mathcal{U}_n$.

Weakly coarse expanding (wxc) systems

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

$$\text{``diam}(\mathcal{U}_n) \to 0\text{''}$$

Weakly coarse expanding (wxc) systems

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.

Weakly coarse expanding (wxc) systems

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.
- **[Irreducibility]**: For any $x \in X$ and any open set $W \ni x$, there exists n such that $f^n(W) \supseteq X$.

Weakly coarse expanding (wxc) systems

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

  $$\text{``diam}(\mathcal{U}_n) \to 0\text{''}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.
- **[Irreducibility]**: For any $x \in X$ and any open set $W \ni x$, there exists n such that $f^n(W) \supseteq X$.
- **[Finiteness]**: B_f is finite

Weakly coarse expanding (wxc) systems

- **[Expanding]**: There exists a cover $\mathcal{U}$ of W_1 such that

$$\text{“diam}(\mathcal{U}_n) \to 0\text{”}$$

  Formally: For any other cover $\mathcal{V}$ there exists N such that for any $n \geq N$, every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{V}$.
- **[Irreducibility]**: For any $x \in X$ and any open set $W \ni x$, there exists n such that $f^n(W) \supseteq X$.
- **[Finiteness]**: B_f is finite

Note: If $W_1 \subseteq S^2$, then [Finiteness] is automatic (Whyburn).

From topological to metric

Definition

A metric ρ on X is exponentially contracting if $\exists C, \alpha > 0$ such that

$$\mathrm{diam}_\rho(U) \leq Ce^{-\alpha n}$$

for any $U \in \mathcal{U}_n$.

From topological to metric

Definition

A metric ρ on X is exponentially contracting if $\exists C, \alpha > 0$ such that

$$\mathrm{diam}_\rho(U) \le Ce^{-\alpha n}$$

for any $U \in \mathcal{U}_n$.

Lemma

For any weakly coarse expanding system $f : W_1 \to W_0$, there exists an exponentially contracting metric ρ on the repellor X.

From topological to metric

Definition

A metric ρ on X is <u>exponentially contracting</u> if $\exists C, \alpha > 0$ such that

$$\mathrm{diam}_\rho(U) \leq Ce^{-\alpha n}$$

for any $U \in \mathcal{U}_n$.

Lemma

For any weakly coarse expanding system $f : W_1 \to W_0$, there exists an exponentially contracting metric ρ on the repellor X.

Definition

A metric ρ on X is a <u>visual metric</u> if $\exists C_1, C_2, \alpha > 0$ such that

$$C_1 e^{-\alpha n} \leq \mathrm{diam}_\rho(U) \leq C_2 e^{-\alpha n}$$

for any $U \in \mathcal{U}_n$.

Examples of weakly coarse expanding systems

- Expanding Thurston maps

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps:

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle.

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

  $$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps:

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

  $$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

  $$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j with $n_j \leq P \cdot j$ and for each j

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j with $n_j \leq P \cdot j$ and for each j

$$\#\{0 \leq i < n_j \ : \ Comp_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \mathrm{Crit}\, f \neq \emptyset\} \leq M$$

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j with $n_j \leq P \cdot j$ and for each j

$$\#\{0 \leq i < n_j \ : \ Comp_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \text{Crit}\, f \neq \emptyset\} \leq M$$

  These are wcx

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

$$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j with $n_j \leq P \cdot j$ and for each j

$$\#\{0 \leq i < n_j \ : \ Comp_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \text{Crit } f \neq \emptyset\} \leq M$$

  These are wcx but not cxc.

Examples of weakly coarse expanding systems

- Expanding Thurston maps
- Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.
- Collet-Eckmann rational maps:

  $$|(f^n)'(f(c))| \geq C\lambda^n$$

  for $\lambda > 1$ if c does not map to another critical point, and there are no parabolic cycles
- Topological Collet-Eckmann rational maps: there exist $M \geq 0, P \geq 1, r > 0$ such that for every $x \in X$ there exists a sequence n_j with $n_j \leq P \cdot j$ and for each j

  $$\#\{0 \leq i < n_j \ : \ Comp_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \text{Crit } f \neq \emptyset\} \leq M$$

  These are wcx but not cxc.

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

with $|\lambda| > 1$.

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

  with $|\lambda| > 1$.
- Semigroups of rational maps (Atnip-Sumi-Urbański)

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

  with $|\lambda| > 1$.
- Semigroups of rational maps (Atnip-Sumi-Urbański)
  Let $f_1, \ldots, f_d$ be rational maps.

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

  with $|\lambda| > 1$.
- Semigroups of rational maps (Atnip-Sumi-Urbański)
  Let $f_1, \ldots, f_d$ be rational maps.

$$F : \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}}$$

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

  with $|\lambda| > 1$.

- Semigroups of rational maps (Atnip-Sumi-Urbański)
  Let $f_1, \ldots, f_d$ be rational maps.

$$F : \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}}$$

$$F(\omega, z) := (\sigma(\omega), f_{\omega_0}(z))$$

Examples of weakly coarse expanding systems - II

- Polymodials (Blokh-Cleveland-Misiurewicz)

$$f(z) = \lambda \frac{z^2}{|z|} + 1$$

  with $|\lambda| > 1$.
- Semigroups of rational maps (Atnip-Sumi-Urbański)
  Let $f_1, \ldots, f_d$ be rational maps.

$$F : \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \{1, \ldots, d\}^{\mathbb{N}} \times \widehat{\mathbb{C}}$$

$$F(\omega, z) := (\sigma(\omega), f_{\omega_0}(z))$$

  Finiteness holds under certain conditions: e.g. if

$$f_1(z) = z^2 + 2, f_2(z) = z^2 - 2$$

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X,

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$:

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$: $P_{top}(\varphi)$ is top. **entropy**,

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**
If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$: $P_{top}(\varphi)$ is top. **entropy**, μ is measure of maximal entropy

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$: $P_{top}(\varphi)$ is top. **entropy**, μ is measure of maximal entropy
- $\varphi = -s \log |f'|$:

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$: $P_{top}(\varphi)$ is top. **entropy**, μ is measure of maximal entropy
- $\varphi = -s \log |f'|$: μ is a conformal measure of dimension s

Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function called a **potential.**

If μ is an f-invariant measure on X, its **pressure** is

$$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$$

and the **topological pressure** is

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} P_\mu(\varphi)$$

Definition

A measure μ which achieves the sup is called an **equilibrium state.**

- $\varphi = 0$: $P_{top}(\varphi)$ is top. **entropy**, μ is measure of maximal entropy
- $\varphi = -s \log |f'|$: μ is a conformal measure of dimension $s \Rightarrow$ compute Hausdorff dimension of X

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$,

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let X be its repellor, ρ an exp. contr. metric on X,

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder.

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then:

1. *there exists a unique equilibrium state μ_φ for φ on X.*

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then:

1. *there exists a unique equilibrium state μ_φ for φ on X.*

   Let $\psi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous observable, and denote

$$S_n\psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x)).$$

Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then:

1. *there exists a unique equilibrium state μ_φ for φ on X.*

   Let $\psi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous observable, and denote

$$S_n\psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x)).$$

   Then there exists the finite limit

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n} \int_X \left(S_n\psi(x) - n \int \psi \, d\mu_\varphi \right)^2 d\mu_\varphi \geq 0$$

   such that the following statistical laws hold:

Statement of results - II

Theorem

1. *(Central Limit Theorem, CLT) For any $a < b$,*

$$\mu_\varphi\left(\left\{x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n}} \in [a,b]\right\}\right) \longrightarrow \frac{\int_a^b e^{-t^2/2\sigma^2}\, dt}{\sqrt{2\pi\sigma^2}}$$

Statement of results - II

Theorem

1. *(Central Limit Theorem, CLT) For any $a < b$,*

$$\mu_\varphi\left(\left\{x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n}} \in [a,b]\right\}\right) \longrightarrow \frac{\int_a^b e^{-t^2/2\sigma^2}\, dt}{\sqrt{2\pi\sigma^2}}$$

If $\sigma = 0$, it converges to a δ-mass at 0.

Statement of results - II

Theorem

1. *(Central Limit Theorem, CLT) For any* $a < b$ *,*

$$\mu_\varphi\left(\left\{x \in X \;:\; \frac{S_n\psi(x) - n\int_X \psi\, d\mu_\varphi}{\sqrt{n}} \in [a,b]\right\}\right) \longrightarrow \frac{\int_a^b e^{-t^2/2\sigma^2}\, dt}{\sqrt{2\pi\sigma^2}}$$

   If $\sigma = 0$*, it converges to a* δ*-mass at* 0*.*

2. *(Law of Iterated Logarithm, LIL) For* μ_φ*-a.e.* $x \in X$*,*

$$\limsup_{n\to\infty} \frac{S_n\psi(x) - n\int_X \psi\, d\mu_\varphi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2}.$$

Statement of results - II

Theorem

1. *(Central Limit Theorem, CLT) For any* $a < b$ *,*

$$\mu_\varphi\left(\left\{x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n}} \in [a, b]\right\}\right) \longrightarrow \frac{\int_a^b e^{-t^2/2\sigma^2}\, dt}{\sqrt{2\pi\sigma^2}}$$

   If $\sigma = 0$*, it converges to a* δ*-mass at* 0*.*

2. *(Law of Iterated Logarithm, LIL) For* μ_φ*-a.e.* $x \in X$*,*

$$\limsup_{n\to\infty} \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2}.$$

3. *(Exponential Decay of Correlations, EDC) For any* $\chi \in L^1(X, \mu_\varphi)$ *there exist* $\alpha > 0, C \geq 0$ *such that*

$$\left|\int_X \psi \cdot (\chi \circ f^n)\, d\mu_\varphi - \int_X \psi \, d\mu_\varphi \cdot \int_X \chi \, d\mu_\varphi\right| \leq Ce^{-n\alpha}.$$

Statement of results - II

Theorem

1. *(Central Limit Theorem, CLT) For any* $a < b$ *,*

$$\mu_\varphi\left(\left\{x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n}} \in [a, b]\right\}\right) \longrightarrow \frac{\int_a^b e^{-t^2/2\sigma^2} \, dt}{\sqrt{2\pi\sigma^2}}$$

If $\sigma = 0$*, it converges to a* δ*-mass at* 0*.*

2. *(Law of Iterated Logarithm, LIL) For* μ_φ*-a.e.* $x \in X$*,*

$$\limsup_{n\to\infty} \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2}.$$

3. *(Exponential Decay of Correlations, EDC) For any* $\chi \in L^1(X, \mu_\varphi)$ *there exist* $\alpha > 0, C \geq 0$ *such that*

$$\left|\int_X \psi \cdot (\chi \circ f^n) \, d\mu_\varphi - \int_X \psi \, d\mu_\varphi \cdot \int_X \chi \, d\mu_\varphi\right| \leq Ce^{-n\alpha}.$$

Statement of results - III

Theorem

1. *(Large Deviations, LD)*

Statement of results - III

Theorem

1. *(Large Deviations, LD) For every $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \Big\{ x \in X : \mathrm{sgn}(t) S_n\psi(x) \geq \mathrm{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \Big\} =$$

Statement of results - III

Theorem

1. *(Large Deviations, LD) For every $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \Big\{ x \in X : \mathrm{sgn}(t) S_n\psi(x) \geq \mathrm{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \Big\} =$$

$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

Statement of results - III

Theorem

1. *(Large Deviations, LD) For every* $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \Big\{ x \in X : \mathrm{sgn}(t) S_n\psi(x) \geq \mathrm{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \Big\} =$$

$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

2. $\sigma = 0$ *if and only if there exists a continuous* $u : X \to \mathbb{R}$ *such that*

$$\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u.$$

Statement of results - III

Theorem

1. *(Large Deviations, LD) For every* $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \Big\{ x \in X : \operatorname{sgn}(t) S_n\psi(x) \geq \operatorname{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \Big\} =$$

$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

2. $\sigma = 0$ *if and only if there exists a continuous* $u : X \to \mathbb{R}$ *such that*

$$\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u.$$

3. $\mu_{\varphi_1} = \mu_{\varphi_2}$ *if and only if there exists* $K \in \mathbb{R}$ *and a continuous* $u : X \to \mathbb{R}$ *such that*

$$\varphi_1 - \varphi_2 = u \circ f - u + K.$$

Statement of results - III

Theorem

1. *(Large Deviations, LD) For every* $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \Big\{ x \in X : \operatorname{sgn}(t) S_n\psi(x) \geq \operatorname{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \Big\} =$$

$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

2. $\sigma = 0$ *if and only if there exists a continuous* $u : X \to \mathbb{R}$ *such that*

$$\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u.$$

3. $\mu_{\varphi_1} = \mu_{\varphi_2}$ *if and only if there exists* $K \in \mathbb{R}$ *and a continuous* $u : X \to \mathbb{R}$ *such that*

$$\varphi_1 - \varphi_2 = u \circ f - u + K.$$

In (2) *and* (3), *u is Hölder continuous w.r.t. the visual metric.*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points,

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X,

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder.

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ *be a wcx system with no periodic critical points, let* X *be its repellor,* ρ *an exp. contr. metric on* X*, and let* $\varphi : (X, \rho) \to \mathbb{R}$ *be Hölder. Then we have:*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then we have:

1. *there exists a unique equilibrium state μ_φ for φ on X.*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then we have:

1. *there exists a unique equilibrium state μ_φ for φ on X.*
2. *Central limit theorem*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then we have:

1. *there exists a unique equilibrium state μ_φ for φ on X.*
2. *Central limit theorem*
3. *Law of Iterated Logarithm*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then we have:

1. *there exists a unique equilibrium state μ_φ for φ on X.*
2. *Central limit theorem*
3. *Law of Iterated Logarithm*
4. *Exponential Decay of Correlations*

Statement of results - IV

Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points, let X be its repellor, ρ an exp. contr. metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder. Then we have:

1. *there exists a unique equilibrium state μ_φ for φ on X.*
2. *Central limit theorem*
3. *Law of Iterated Logarithm*
4. *Exponential Decay of Correlations*
5. *Large Deviations*

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d.

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift.

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

such that

$$\pi \circ \sigma = f \circ \pi$$

Proof: the geometric coding

Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \dots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

such that

$$\pi \circ \sigma = f \circ \pi$$

Idea: geometric coding

Proof: the geometric coding

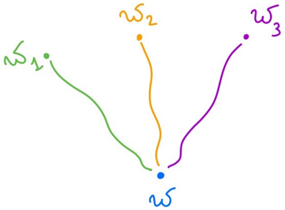
Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \dots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

such that

$$\pi \circ \sigma = f \circ \pi$$

Idea: geometric coding



Proof: the geometric coding

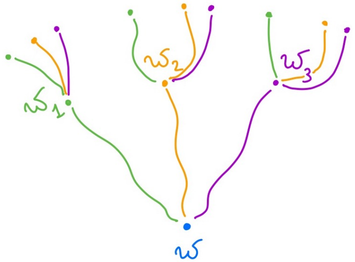
Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

such that

$$\pi \circ \sigma = f \circ \pi$$

Idea: geometric coding



Proof: the geometric coding

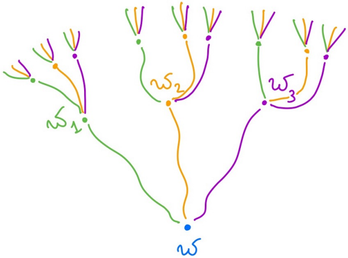
Let $f : W_1 \to W_0$ be wcx of degree d. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ the shift. Then there exists a semiconjugacy

$$\pi : \Sigma \to X$$

such that

$$\pi \circ \sigma = f \circ \pi$$

Idea: geometric coding



Proof: no periodic critical points $\rightarrow$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_* \mu}(f)$$

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_*\mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Proof: no periodic critical points $\rightarrow$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_*\mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{i \leq n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \leq \zeta n$$

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_* \mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{i \leq n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \leq \zeta n$$

Lemma $\Rightarrow$

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_* \mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{ i \leq n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \leq \zeta n$$

Lemma $\Rightarrow S_{n,x} := \#\{\text{n-cylinders intersecting } \pi^{-1}(x)\}$

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_* \mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{i \leq n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \leq \zeta n$$

Lemma $\Rightarrow S_{n,x} := \#\{\text{n-cylinders intersecting } \pi^{-1}(x)\}$

$$\lim_{n \to \infty} \frac{1}{n} \log S_{n,x} = 0$$

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_*\mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{i \le n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \le \zeta n$$

Lemma $\Rightarrow S_{n,x} := \#\{\text{n-cylinders intersecting } \pi^{-1}(x)\}$

$$\lim_{n\to\infty} \frac{1}{n} \log S_{n,x} = 0$$

"fiber is subexponentially small"

Proof: no periodic critical points $\to$ no entropy drop

Lemma (No entropy drop)

For any invariant measure μ, we have

$$h_\mu(\sigma) = h_{\sigma_* \mu}(f)$$

For a finite orbit (x_i), we call an ϵ-singular time an index i s.t. exists y with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$.

Lemma

For any $0 < \zeta < 1$ there exists ϵ s.t.

$$\#\{i \le n \ : \ x_i \text{ is } \epsilon - \text{singular } \} \le \zeta n$$

Lemma $\Rightarrow$ $S_{n,x} := \#\{\text{n-cylinders intersecting } \pi^{-1}(x)\}$

$$\lim_{n\to\infty} \frac{1}{n} \log S_{n,x} = 0$$

“fiber is subexponentially small” $\to$ no entropy drop

Periodic critical points

Warning: for periodic critical points entropy may drop for some measure μ.

The blowup procedure

Suppose $f(p) = p$ is a critical point.

The blowup procedure

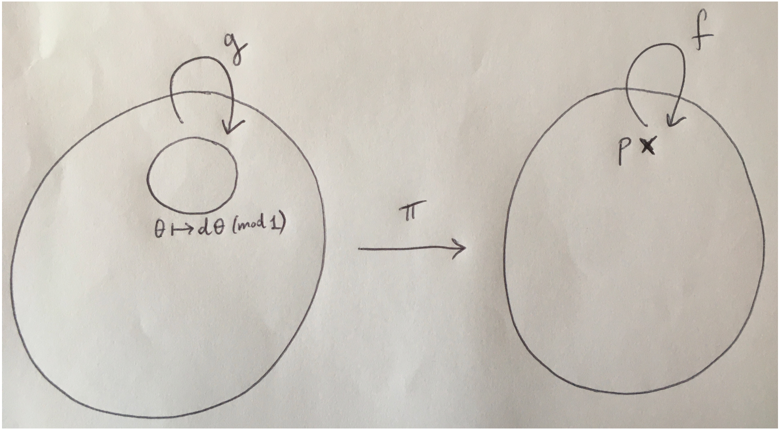
Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$

The blowup procedure

Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle

The blowup procedure

Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle



The blowup procedure

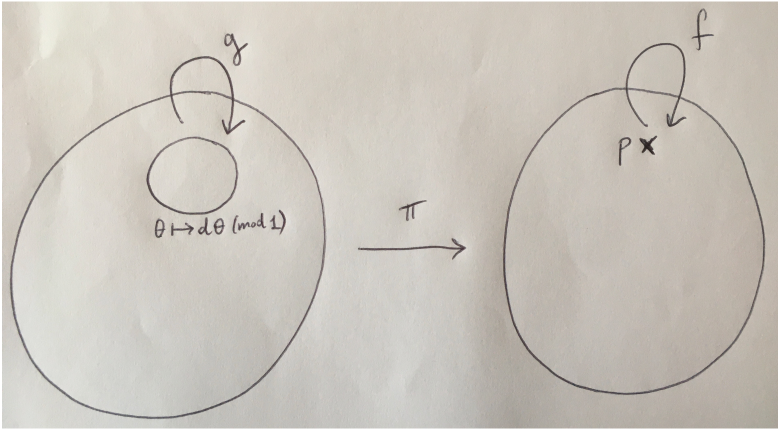
Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$

The blowup procedure

Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$.

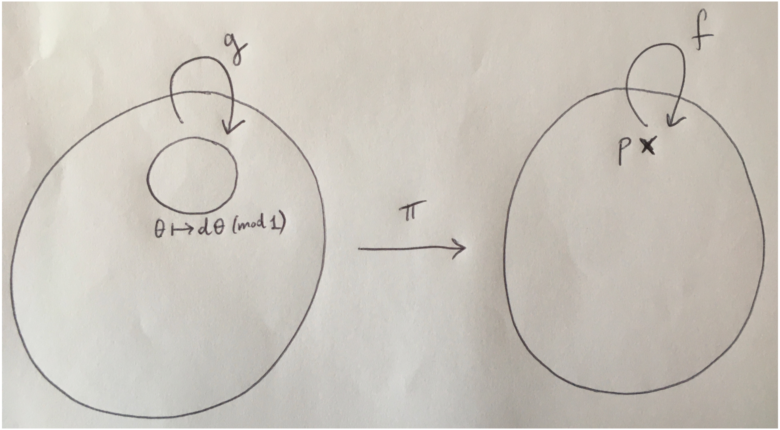
The blowup procedure

Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$.



The blowup procedure

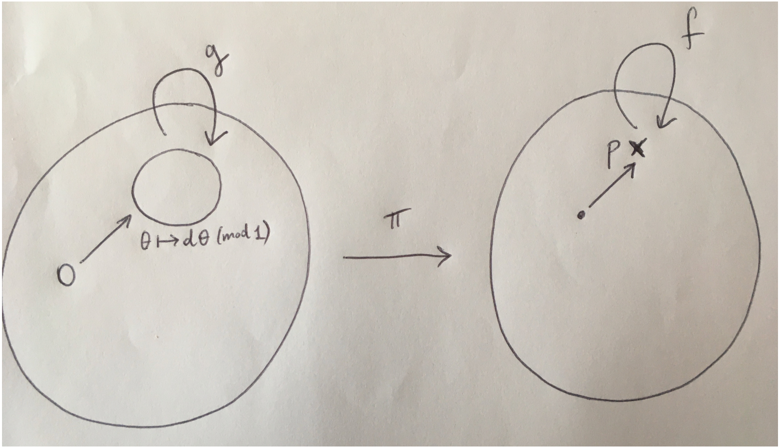
Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$.



$Y \cong$ Siérpinski carpet $\to$ $X \cong$ sphere

The blowup procedure

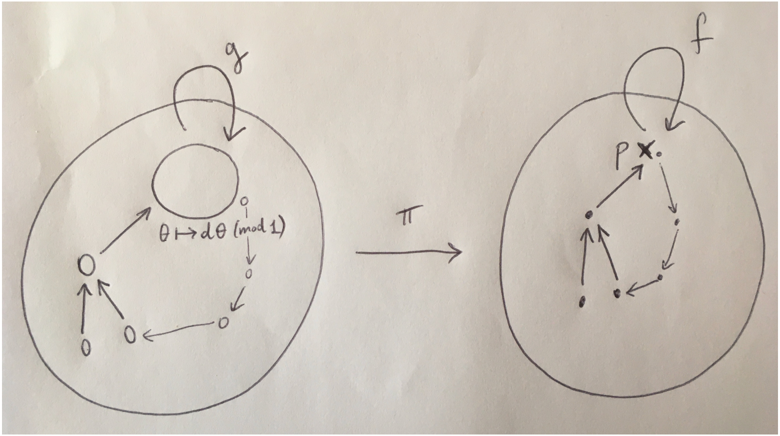
Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$.
Extend the dynamics on the preimage of the critical orbit.



$Y \cong$ Siérpinski carpet $\quad\to\quad$ $X \cong$ sphere

The blowup procedure

Suppose $f(p) = p$ is a critical point.
Construct $\pi : Y \to X$ by blowing up critical point to a circle
Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$.
Extend the dynamics on the preimage of the critical orbit.



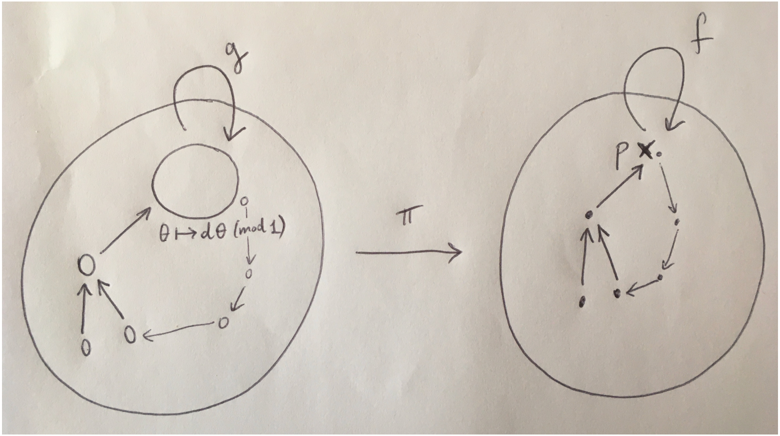
$Y \cong$ Siérpinski carpet $\quad \to \quad X \cong$ sphere

The blowup procedure

Hard part: define a metric $\widetilde{\rho}$ on Y which is exponentially contracting for g.

The blowup procedure

Hard part: define a metric $\widetilde{\rho}$ on Y which is exponentially contracting for g.



The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding,

The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding, one constructs semiconjugacies:
$$\Sigma \to Y \to X$$

The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding, one constructs semiconjugacies:
$$\Sigma \to Y \to X$$

- Then $g : Y \to Y$ is wcx without periodic critical points

The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding, one constructs semiconjugacies:
$$\Sigma \to Y \to X$$

- Then $g : Y \to Y$ is wcx without periodic critical points hence it satisfies the hypotheses of the first version of the theorem.

The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding, one constructs semiconjugacies:
$$\Sigma \to Y \to X$$
- Then $g : Y \to Y$ is wcx without periodic critical points hence it satisfies the hypotheses of the first version of the theorem.
- Then: for any equilibrium measure μ on Σ, there is no entropy drop.

The blowup procedure

- Since $(Y, \widetilde{\rho})$ is weakly coarse expanding, one constructs semiconjugacies:
$$\Sigma \to Y \to X$$
- Then $g : Y \to Y$ is wcx without periodic critical points hence it satisfies the hypotheses of the first version of the theorem.
- Then: for any equilibrium measure μ on Σ, there is no entropy drop.
- The general theorem follows.

Further directions

- Do you have equidistribution of preimages?

Further directions

- Do you have equidistribution of preimages? (Z. Li)

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)?

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)? Then you need to consider the geometric potential $\log |f'|$ which is not Hölder

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)? Then you need to consider the geometric potential $\log|f'|$ which is not Hölder (for rational maps: F. Przytycki, J. Rivera Letelier, ...)

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)? Then you need to consider the geometric potential $\log |f'|$ which is not Hölder (for rational maps: F. Przytycki, J. Rivera Letelier, ...)
- Do you have a prime number theorem, i.e. counting periodic points according to period and/or multiplier?

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\# f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)? Then you need to consider the geometric potential $\log |f'|$ which is not Hölder (for rational maps: F. Przytycki, J. Rivera Letelier, ...)
- Do you have a prime number theorem, i.e. counting periodic points according to period and/or multiplier? (Z. Li-T. Zheng)

Further directions

- Do you have equidistribution of preimages? (Z. Li)

$$\frac{1}{\#f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu$$

- Can you extract geometric information on the repeller X (i.e., its Hausdorff dimension)? Then you need to consider the geometric potential $\log |f'|$ which is not Hölder (for rational maps: F. Przytycki, J. Rivera Letelier, ...)
- Do you have a prime number theorem, i.e. counting periodic points according to period and/or multiplier? (Z. Li-T. Zheng)
- (D. Meyer) Can you identify the Lebesgue measure from the thermodynamics?

The end

Thank you!