RESEARCH STATEMENT

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My research focuses on dynamical systems and their relations with complex analysis, Teichmüller theory, ergodic theory, probability and group theory. My two main lines of research are the following.

(1) One-dimensional real and complex dynamics

I study in particular the *core entropy* of complex quadratic polynomials and how it behaves as a function of the Mandelbrot set. The core entropy is a new invariant, introduced by W. Thurston in one of his last projects, and I developed part of this theory, building on ideas of Thurston, and solving some of his conjectures. One of the main goals is to relate the core entropy function to the geometry of the Mandelbrot set.

(2) Random walks on groups

I look at asymptotic properties of random walks on spaces which have, to a certain extent, negative curvature geometry. In particular, I proved a multiplicative ergodic theorem for random walks on the mapping class group which solves a conjecture of Kaimanovich, and proved that the hitting measures for certain random walks on $Mod(S)$ are singular at the boundary of Teichmüller space. Maher and I worked out a general theory of random walks on groups acting on Gromov hyperbolic spaces, which has various applications to geometric group theory.

Further projects are related to the dynamics of a family of continued fraction transformations, the dimension of attractors for circle maps, and generalized analytic continuation. However, we will only describe (1) and (2) in detail here.

1. Entropy of complex polynomials

A central theme in dynamical systems is the variation of dynamics along families. Remarkably, the qualitative aspects of the dynamics may change abruptly as parameters change, and typically, in parameter space there are infinitely many stability islands, which are intertwined with chaotic regions. Moreover, the local geometry of parameter space often reflects the dynamics of the underlying dynamical system.

An example of this phenomenon arises in the real quadratic family $f_c(z) := z^2 + c$, with $c \in [-2, 1/4]$. Each such f_c preserves an interval, so one can consider the topological entropy $h(c) := h_{top}(f_c)$ of this interval map. As proven by Milnor-Thurston [MT], the entropy $h(c)$ is a continuous, monotone function of the parameter c, whose graph displays fractal features (Figure 2, left).

1.1. The core entropy.
In order to generalize this theory to complex polynomials, around 2011 W. Thurston defined a new dynamical invariant, the *core entropy*. Recall a polynomial $f : \mathbb{C} \to \mathbb{C}$ is *critically finite* if the forward orbit of every critical point of f is finite. Then the filled Julia set of f contains a forward invariant, finite topological tree, called the *Hubbard tree* [DH1] (see Figure 3, left).

Definition 1.1. *The* core entropy *of f is the topological entropy of the restriction of f to its Hubbard tree.*

The central object in the parameter space of quadratic polynomials is the *Mandelbrot set* $\mathcal{M}$, which is the set of parameters $c \in \mathbb{C}$ for which the map f_c has bounded critical orbit. Critically finite quadratic polynomials are parameterized by rational numbers: given $\theta \in \mathbb{Q}/\mathbb{Z}$, the external ray at angle θ determines a critically finite parameter c_θ in the Mandelbrot set [DH1]. We define $h(\theta)$ to be the core entropy of $f_\theta(z) := z^2 + c_\theta$. In [Ti5], I show:

Theorem 1.2 ([Ti5]). *The core entropy function $h : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ extends to a continuous function $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$.*

The theorem proves a conjecture of W. Thurston, who first explored the parameter spaces of polynomials in terms of core entropy [Th+]. As he showed, the core entropy function can be defined purely combinatorially, but it displays a rich fractal structure (Figure 1), which reflects the underlying geometry of the Mandelbrot set.

Hubbard trees have been introduced to classify critically finite maps up to topological conjugacy ([DH1], [Po]), and their entropy provides a new tool to study the parameter space of polynomials: for instance, the restriction of $h(\theta)$ to veins of the Mandelbrot set is also monotone ([Li], [Ze]) which implies that the lamination for the Mandelbrot set can be reconstructed by looking at level sets of h. Note by comparison that the entropy of f_θ on its Julia set is equal to $\log 2$, independently of θ, hence it does not give information on the parameter.
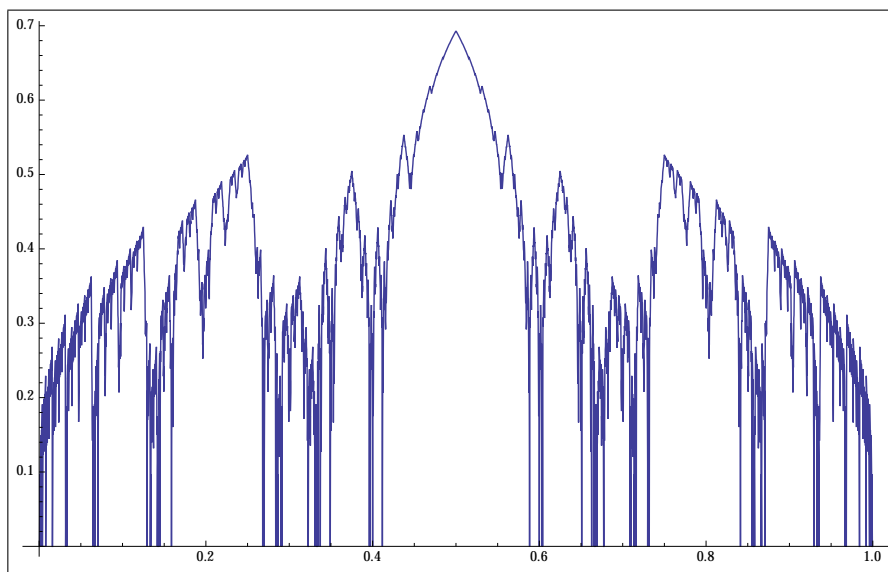

FIGURE 1. The core entropy function $h(\theta)$ as a function of the external angle θ. Can you see the Mandelbrot set in this picture?

Once continuity is established, one can use the core entropy to study the parameter space of quadratic polynomials. In particular, we are interested in **relating the core entropy function to the geometry of the Mandelbrot set**.

1.2. **Entropy and biaccessibility.** The core entropy is also related to the widely-studied set B_c of *bi-accessible angles* in the Julia set of f_c (see [Sm], [Zd], [MS], [BS]), i.e. angles s such that
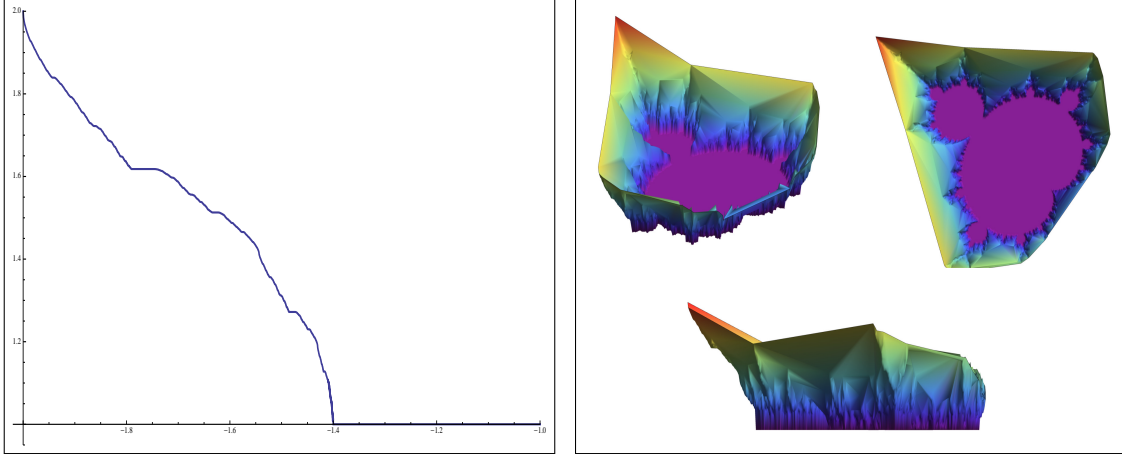
FIGURE 2. Left: the topological entropy $h(c)$ of real quadratic polynomials, as a function of c [MT]. Right: Three views of the graph of the core entropy $h(c)$ as a function of c. Note in the side view the profile of the Milnor-Thurston real entropy picture.

there exists some other angle $t \neq s$ such that the external rays at angle s and at angle t land together in the Julia set of f_c. We call c a *topologically finite* parameter if the Hubbard tree of f_c is homeomorphic to a finite tree. Bruin-Schleicher and I independently showed:

Theorem 1.3 ([Ti3], [BS]). *Let c be a topologically finite parameter. Then we have the equality*

$$\text{H.dim } B_c = \frac{h(c)}{\log 2}.$$

The result is in line with the classical "entropy formula" [Man] relating dimension, entropy and Lyapunov exponents (as $\log 2$ is the Lyapunov exponent of the doubling map).

1.3. Entropy and Hausdorff dimension along veins.

The folklore *Douady principle* states that the local geometry of the Mandelbrot set near a parameter c reflects the geometry of the Julia set $J(f_c)$ for the corresponding map. In [Ti3], I establish an instance of this principle by looking at the Hausdorff dimension of certain sets of angles.

Recall that a *vein* in the Mandelbrot set $\mathcal{M}$ is a continuous, injective arc joining 0 to c, which we denote as $[0, c]$ (see [BD]). The Riemann mapping $\Phi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \mathcal{M}$ defines a coordinate system on the exterior of the Mandelbrot set, and given $\theta \in \mathbb{R}/\mathbb{Z}$, the image $R_M(\theta) := \Phi(\{re^{2\pi i\theta}, r > 1\})$ of the radial arc at angle θ is called the *external ray* of angle θ. The ray is said to *land* at $x \in \partial\mathcal{M}$ if $\lim_{r \to 1+} \Phi(re^{2\pi i\theta}) = x$. Moreover, given a subset A of $\mathcal{M}$, we define

$$\Theta(A) := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ R_M(\theta) \text{ lands at } A\}$$

to be the set of angles of rays which land on A. Given a subset A of $\partial\mathcal{M}$, the *harmonic measure* of A is the Lebesgue measure of $\Theta(A)$, which can be interpreted as the probability that a random ray from infinity lands on A. The harmonic measure of the *real slice* $\partial\mathcal{M} \cap \mathbb{R}$ of the Mandelbrot set is zero; however, the set $\mathcal{R} = \Theta(\mathbb{R})$ of angles corresponding to rays landing on the real axis has Hausdorff dimension 1 [Za]. I relate the Hausdorff dimension of the angles landing on the vein to the core entropy:

Theorem 1.4 ([Ti3])**.** *Let B_c denote the set of bi-accessible angles in the Julia set of f_c. For any c which belongs to a principal vein in the Mandelbrot set, we have the equality*

$$\text{H.dim } \Theta([0, c]) = \text{H.dim } B_c.$$

Note that the left-hand side arises from the geometry of parameter space, while the right-hand side describes a geometric property of the Julia set of f_c.

The above statement is already new for real parameters c. In fact, the above mentioned result of Zakeri states that the Hausdorff dimension of the set of external angles which land on the real axis is 1: this is a particular case of Theorem 1.4 for $c = -2$, while my result is true for *any* parameter along the vein. As a corollary, I also answer some questions of Zakeri ([Za], Remark 6.9): namely, the set of biaccessible angles for the Feigenbaum parameter c_{Feig} has dimension 0, and we can improve the bounds on the dimension for parameters $-1.75 \leq c \leq c_{Feig}$. The theorem was later generalized by Jung [Ju] to arbitrary veins.
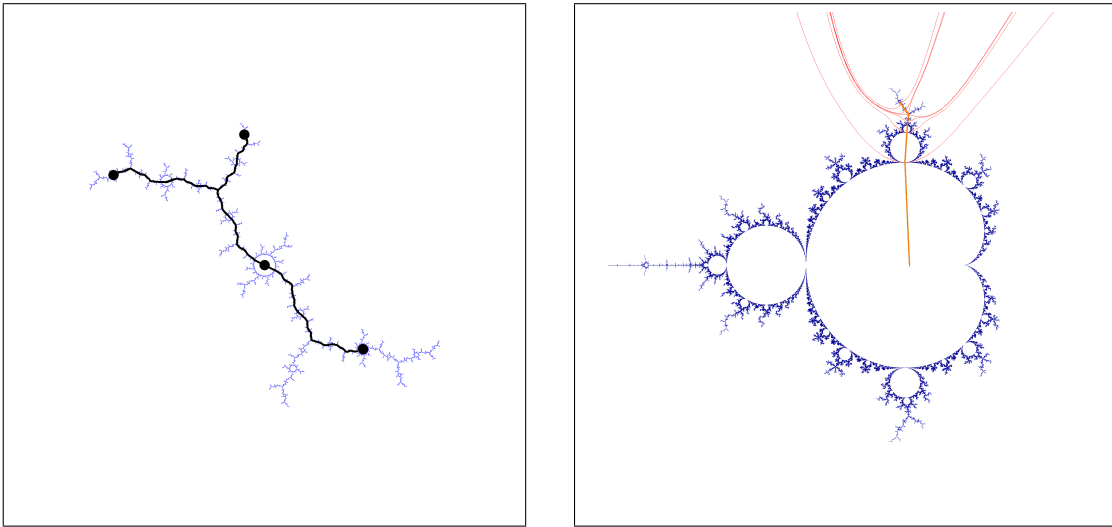

FIGURE 3. To the left: the Hubbard tree for the map f_θ with $\theta = 1/5$; the critical point has period 4. To the right: the external rays landing on the corresponding vein.

1.4. **The infinite clique polynomial.** One of the tools in the proof of Theorem 1.2 is the study of the growth of infinite graphs, which arise as transition graphs from the dynamics on the Hubbard tree. This result can be of independent interest in graph theory. In fact, it is known that one can write the characteristic polynomial of a finite graph in terms of its *clique polynomial*, as used e.g. in [Mc], and I generalize this theory to infinite graphs.

Let Γ be a directed graph with countably many vertices, and suppose the outgoing degree of each vertex is uniformly bounded. Then we say that the graph has *bounded cycles* if for any n, it has finitely many cycles of length n. Then it is well-defined the power series

$$(1) \qquad\qquad P(t) := \sum_\gamma (-1)^{C(\gamma)} t^{\ell(\gamma)}$$

where the sum runs over all simple multi-cycles γ in Γ, $C(\gamma)$ denotes the number of connected components of the multi-cycle γ and $\ell(\gamma)$ its total length. We then define the growth rate of Γ as

$$r(\Gamma) := \limsup_{n \to \infty} \sqrt[n]{\# \text{ closed paths of length } n}$$

and similarly σ as the exponential growth rate of the number of simple multi-cycles of length n. (In the dynamical case, the entropy equals the logarithm of $r(\Gamma)$.) I prove the

Theorem 1.5 ([Ti5]). *Let r be the growth rate of Γ, and consider the power series $P(t)$ defined in (1). If $\sigma \leq 1$, then the function $P(t)$ converges in the unit disk, defining a holomorphic function. Then the smallest real zero of $P(t)$ equals r^{-1}.*

This result generalizes to tree maps one of the main results of the *kneading theory* of Milnor-Thurston; indeed, for any real quadratic polynomial of entropy $h = \log \lambda$, [MT] construct a *kneading series*, i.e. a power series whose smallest real zero is $1/\lambda$. The power series $P(t)$ above plays the role of the kneading series for complex maps.

1.5. **Entropy and Galois conjugates.** We now turn to a discussion of the algebraic properties of entropies, and their relationship with the dynamics. If a dynamical system $f : X \to X$ has a Markov partition, then its topological entropy $h(f)$ is the logarithm of an algebraic number: in fact, if we call *growth rate* of f the quantity

$$s(f) := e^{h(f)}$$

then $s(f)$ is the leading eigenvalue of the transition matrix associated to the partition. By the Perron-Frobenius theorem it follows immediately that $s(f)$ must be a *weak Perron number*, i.e. a real algebraic integer which is at least as large as the modulus of all its Galois conjugates. In [Th2], Thurston asked the converse question, namely what algebraic integers arise as growth rates of dynamical systems with a Markov partition. The question makes sense in several contexts (e.g. for pseudo-Anosov maps of surfaces, as well as automorphisms of the free group), but we will now focus on critically finite real polynomial maps. Thurston proved the following:

Theorem 1.6 ([Th2]). *The set of all growth rates of critically finite real polynomial maps coincides with the set of all weak Perron numbers.*

The question becomes more subtle when one restricts oneself to maps of a given degree. In particular, in the case of degree two, Thurston looked at the algebraic properties of growth rates of critically finite real quadratic polynomials; remarkably, he found out that the union of all their Galois conjugates exhibits a rich structure (Figure 4, left). In order to construct the set, let us call a quadratic polynomial *superattracting* if its critical point is periodic.

Definition 1.7. *The* entropy spectrum Σ *is the closure of the set of Galois conjugates of growth rates of superattracting real quadratic polynomials.*

Thurston claimed (without proof) that such a fractal set is path-connected. I show:

Theorem 1.8 ([Ti4]). *The entropy spectrum Σ is path-connected and locally connected.*

Several people ([BH], [Bo], [Ba], [So]) have studied sets of zeros of polynomials with restricted coefficients, in particular the set $\Sigma^{\pm 1} := \overline{\{z \in \mathbb{C} \ : \ \sum_{i=1}^{n} \epsilon_k z^k = 0, \ \epsilon_i \in \{\pm 1\}\}}$ of roots of all polynomials with coefficients in $\{\pm 1\}$. As we can see from Figure 4, the two sets look similar. I prove that their intersections with the unit disk $\mathbb{D} := \{|z| < 1\}$ actually coincide:
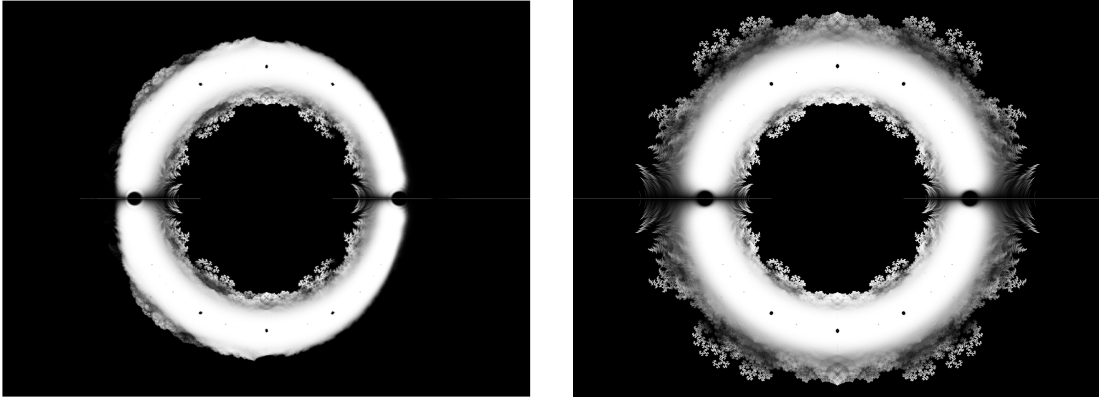
FIGURE 4. Left: the entropy spectrum Σ for real quadratic polynomials. Right: the set $\Sigma^{\pm 1}$ of roots of polynomials with coefficients ± 1.

Theorem 1.9 ([Ti4]). *We have the equality*

$$\Sigma \cap \mathbb{D} = \Sigma^{\pm 1} \cap \mathbb{D}.$$

I constructed analogous fractal sets for each vein in the Mandelbrot set, and also by considering all core entropies of all hyperbolic components at once[1], and I plan to further investigate their geometry.

2. RANDOM WALKS ON GROUPS

A vast area of research in the geometric study of groups has been dedicated to the definition of boundaries on which the group acts, following the example of Fuchsian groups acting on the boundary of hyperbolic plane. Starting from Furstenberg [Fu], one approach to the study of group actions is the probabilistic one, namely via *random walks on groups*. Indeed, let G be a group acting isometrically on a geodesic metric space (X, d), and μ a probability distribution on G. A random walk on G is defined by drawing each time independently an element g_n from G with distribution μ, and considering the product

$$w_n := g_1 \ldots g_n.$$

If we fix a basepoint x in X, the sequence $\{w_n x\}_{n \in \mathbb{N}}$ is a stochastic process with values in X, so we can think of it as a random walk on X. In this setting, we can ask the following basic questions:

- Does a typical sample path escape to infinity?
- Does it converge to some point in a suitable boundary of X?
- Can a typical sample path be approximated by a geodesic in X?

We will be particularly interested in groups acting on spaces with certain negative curvature geometry. One important such example is the case where $G = Mod(S)$ is the *mapping class group*, i.e. the group of isotopy classes of homeomorphisms of a surface S, acting on the *Teichmüller space* $X = \mathcal{T}(S)$, which is a geodesic metric space when equipped with the *Teichmüller metric d_T*. Teichmüller space is not negatively curved, but shares common features with hyperbolic spaces, and many authors (see e.g. [Ma], [Ra]) have addressed the question of up to what extent Teichmüller space is hyperbolic.

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[1]Pictures are available on my website `http://users.math.yale.edu/users/tiozzo/gallerynew.html`.

2.1. **Sublinear tracking: a multiplicative ergodic theorem.** My first result is that random walks with finite first moment on mapping class groups track Teichmüller geodesics sublinearly:

Theorem 2.1 ([Ti2]). *Let μ be a probability measure on $Mod(S)$ with finite first moment, whose support generates a non-elementary group, and fix some x in the Teichmüller space $\mathcal{T}(S)$. Then, there exists $A > 0$ such that for almost all sample paths there exists a Teichmüller geodesic ray $\gamma : [0, \infty) \to \mathcal{T}(S)$ with $\gamma(0) = x$ and such that*

$$\lim_{n \to \infty} \frac{d_T(w_n x, \gamma(An))}{n} = 0.$$

The theorem answers a question of Kaimanovich [Ka00] which has been open for more than a decade. A partial result in this direction was Duchin's thesis [Du] where sublinear tracking is proven along subsequences of times for which the limit geodesic lies in the thick part.

Let us note that, in the simple case of $\mathbb{R}$ acting on itself by translations, a random walk $Y_n := X_1 + \cdots + X_n$ escapes to infinity if and only if its drift $\ell := \mathbb{E}[X_n]$ is non-zero. If that is the case, by the *law of large numbers* almost every sample path escapes linearly, which is equivalent to say that the walk Y_n lies within sublinear distance from the geodesic $\gamma(t) := t$:

$$\frac{|X_1 + \cdots + X_n - \gamma(\ell n)|}{n} \to 0.$$

Thus, sublinear tracking can be seen as a generalization of the law of large numbers to non-commutative groups. The theory of random products on non-abelian groups goes back to the *multiplicative ergodic theorems* of Furstenberg-Kesten [FK] and Oseledets [O]. In the 80's, Kaimanovich [Ka87] realized that the multiplicative ergodic theorem is equivalent to sublinear tracking for random walks on the symmetric space $X = GL(n, \mathbb{R})/O(n, \mathbb{R})$ and proved it for general symmetric spaces of noncompact type, and word hyperbolic groups [Ka94]. Karlsson and Margulis [KM] proved sublinear tracking in non-positively curved ($CAT(0)$) spaces; however, Teichmüller space does not satisfy this property. Another reason of interest in the question is the criterion [Ka85] that sublinear tracking allows one to identify the *Poisson boundary* of the random walk (see also Section 2.3).

The proof of Theorem 2.1 relies only on ergodic theory and on some geometric properties of the boundary, hence it is applicable to a wider class of compactifications, in particular:

(1) the *hyperbolic compactification* of Gromov hyperbolic spaces;
(2) the *end compactification* of Freudenthal and Hopf;
(3) the *Floyd compactification*;
(4) the *visual compactification* of a large class of CAT(0) spaces.

As long as these boundaries are non-trivial, my argument yields sublinear tracking in all these spaces. This includes the case of groups with infinitely many ends, as well as non-elementary Kleinian groups and relatively hyperbolic groups.

2.2. **Random walks on Gromov hyperbolic spaces.** A generalization of spaces of negative curvature is given by Gromov's theory of δ-hyperbolic spaces. Let G be a group of isometries of a δ-hyperbolic, metric space X. Then X is equipped with its *hyperbolic boundary ∂X* (or *Gromov boundary*), and one can ask the following questions:

- Does a generic sample path converge to ∂X?
- Does the random walk have positive speed?
- Is the boundary ∂X with the hitting measure a model for the Poisson boundary of the walk?

The classical theory of random walks on hyperbolic groups was developed by Kaimanovich [Ka94] and others, under the hypothesis that the space X is *proper*, i.e. closed balls in X are compact. However, many groups of interest in topology and geometric group theory act on non-proper δ-hyperbolic spaces (e.g., a locally infinite graph). In particular:

(1) the mapping class group $Mod(S)$ acting on the curve complex $C(S)$;
(2) the group $Out(F_N)$ of outer automorphisms of the free group acting on the complex of free factors or the complex of free splittings;
(3) right-angled Artin groups acting on their extension graph;
(4) the Cremona group acting on the Picard-Manin space.

Maher and I develop the theory of random walks for actions on general (non-proper) δ-hyperbolic spaces, answering the questions above.

Theorem 2.2 ([MaTi])**.** *Let G be a group of isometries of a separable, δ-hyperbolic space X, and let μ be a non-elementary measure on G. Then:*

(1) *The random walk converges almost surely to the Gromov boundary: for almost every sample path (w_n), there exists $\xi \in \partial X$ such that $w_n x \to \xi$.*
(2) *The random walk has positive drift: there exists $L > 0$ such that*

$$\liminf_{n \to \infty} \frac{d(w_n x, x)}{n} \geq L > 0.$$

If μ has finite first moment, then

$$\lim_{n \to \infty} \frac{d(w_n x, x)}{n} = L > 0.$$

(3) *The translation length of random elements grows at least linearly:*

$$\mathbb{P}(\tau(w_n) \geq Ln) \to 1 \qquad \text{as } n \to \infty.$$

As a consequence of (3), the probability that a random group element acts hyperbolically (i.e. with positive translation length) tends to 1 as $n \to \infty$. This generalizes works of Maher [Ma2] and Rivin [Ri1] that random mapping classes are pseudo-Anosov, as well as work of Kapovich-Rivin [Ri2] that random elements of $Out(F_N)$ are atoroidal and fully irreducible.

Note that, since in the non-proper case the boundary ∂X need not be compact, then the issue of convergence is quite delicate: we recur to using the *horofunction compactification* (as in [KL]), which is always compact, and then relate this to the Gromov boundary.

2.3. **The Poisson boundary for acylindrical actions.** The *Poisson boundary* of a random walk (G, μ) is a canonical measure space (B, ν) which induces an isomorphism between the space of bounded μ-harmonic functions on G and the space $L^\infty(B, \nu)$ of bounded ν-measurable functions on B. If the space X has a natural topological boundary ∂X, many authors have asked whether ∂X with the hitting measure ν is a model for the Poisson boundary. Maher and I prove it for groups acting acylindrically on Gromov hyperbolic spaces.

Theorem 2.3 ([MaTi])**.** *Let μ be a measure on G with finite entropy and finite logarithmic moment, and suppose that the action of G on X is acylindrical. Then the Gromov boundary $(\partial X, \nu)$ is a model for the Poisson boundary.*

Acylindricality is a weak form of properness of the action, originally due to Sela and now established for many groups of interest (see e.g. [Os]): it means that for any K there exist R, N such that for any pair of points at distance larger than R, the K-coarse stabilizer of the pair has cardinality

at most N. As the action of the mapping class group on the curve complex is acylindrical [Bow], we prove:

Corollary 2.4 ([MaTi]). *The Gromov boundary of the curve complex is a model for the Poisson boundary of the mapping class group.*

Kaimanovich-Masur [KaM] identified the Poisson boundary of the mapping class group with the (Thurston-) boundary of Teichmüller space. In our approach, we determine the Poisson boundary of the mapping class group in a purely topological way, without resorting to Teichmüller theory.

2.4. **Random subgroups and random group extensions.** The previous results also have applications in terms of random subgroups and random group extensions. Following Guivarc'h [Gu] and Rivin [Ri2], we define a model of random subgroups as follows. Given $k \geq 1$, let us consider k independent random walks on G, all with the same distribution μ, and consider the subgroup of G generated by them. Taylor and I prove that random k-generated subgroups are almost surely free and undistorted in X:

Theorem 2.5 ([TT]). *Let $w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(k)}$ be k independent random walks on G, and consider*

$$\Gamma_n := \langle w_n^{(1)}, \ldots, w_n^{(k)} \rangle$$

the subgroup generated by them. Then the probability that Γ_n is free and quasi-isometrically embeds in X tends to 1 as $n \to \infty$.

As a corollary, using the action of the mapping class group on the curve complex, we show:

Corollary 2.6 ([TT]). *Random k-generated subgroups of the mapping class group are almost surely convex cocompact.*

We also prove an analogous statement for random subgroups of $Out(F_N)$. This also yields a model for random extensions of free and surface groups. In fact, by the Birman exact sequence

$$1 \to \pi_1(S) \to Mod(S; p) \to Mod(S) \to 1,$$

the preimage of any subgroup Γ_n by the forgetful map $Mod(S; p) \to Mod(S)$ is an extension of the surface group $\pi_1(S)$, and the group is known to be hyperbolic iff Γ_n is convex cocompact. Thus, we show:

Corollary 2.7 ([TT]). *Random extensions of surface groups are hyperbolic. Similarly, random extensions of free groups are hyperbolic.*

2.5. **Harmonic measure and singularity.** In the context of Lie groups, Furstenberg [Fu] famously constructed, for any lattice in a semisimple Lie group, a random walk on the lattice such that its harmonic measure is the Lebesgue measure on the boundary. This is the first step in the proofs of the rigidity theorems by Furstenberg and Margulis [Mar].

One wants to extend the theory to the case of the *Thurston boundary* $\mathcal{PMF}$ of Teichmüller space, which is naturally endowed with a *Lebesgue measure class*, and we want to compare it to the harmonic measure. A long-standing question is the following:

Question 2.8 ([KaM]). *Is there a measure on the mapping class group such that the hitting measure of its corresponding random walk is equivalent to Lebesgue measure?*

In [GMT], we address this question by **relating the singularity problem for random walks on the mapping class group to the behavior of generic Teichmüller geodesics**. Indeed, the quotient $\mathcal{T}(S)/Mod(S)$ is the moduli space of Riemann surfaces, which is a non-compact orbifold

of finite volume, and we can study excursions of typical geodesics into the *thin part* $\mathcal{M}_\epsilon$ of moduli space, which plays the role of the cusp of a hyperbolic manifold. The idea is that a typical geodesic with respect to the Lebesgue measure will penetrate more deeply into the cusp than a typical geodesic with respect to harmonic measure, so the measures are mutually singular.

The setup is the following. Let us fix a basepoint X in Teichmüller space, let F be a foliation on the Thurston boundary, and let γ_t be the Teichmüller geodesic ray based at X with vertical foliation F. For each t, denote as g_t a mapping class such that $g_t X$ is the closest element to γ_t in the orbit of X. This way, we associate to each geodesic a sequence of group elements, and we want to study the growth rate of the sequence as a function of t. Gadre, Maher and I prove that the ratio of the two metrics tends to infinity for Lebesgue-typical geodesics, while it stays bounded for generic geodesics with respect to the harmonic measure:

Theorem 2.9 ([GMT]). *Let μ be a measure on $Mod(S)$ of finite first moment in the word metric, and such that the group generated by its support is non-elementary, and let ν be its harmonic measure on $\mathcal{PMF}$. Then:*

(1) *For almost every foliation $F \in \mathcal{PMF}$ with respect to the Lebesgue measure class,*

$$\lim_{\substack{t \to \infty \\ \gamma(t) \notin \mathcal{M}_\epsilon}} \frac{\|g_t\|_G}{t} = +\infty.$$

(2) *There exists a constant c (depending on μ) such that, for almost every foliation $F \in \mathcal{PMF}$ with respect to the harmonic measure ν,*

$$\lim_{\substack{t \to \infty \\ \gamma(t) \notin \mathcal{M}_\epsilon}} \frac{\|g_t\|_G}{t} = c.$$

Corollary 2.10 ([GMT]). *Let μ be a measure of finite first moment in the word metric on the mapping class group. Then the harmonic measure for the associated random walk is singular with respect to the Lebesgue measure class.*

This implies that any measure which answers Question 2.8 cannot have finite first moment in the word metric. However, the question is still open for general measures, and we plan to address it. Even more interesting would be to have an example of a measure of finite first moment *in the Teichmüller metric* whose harmonic measure lies in the Lebesgue measure class. Further extensions of this approach we plan to pursue are:

- prove singularity of the harmonic measure for random walks on discrete groups of isometries of higher dimensional hyperbolic space $\mathbb{H}^n$ such that the quotient manifold has a cusp;
- study the quotient $\frac{\|g_t\|_G}{t}$ along generic geodesics for the *Weil-Petersson* metric (note that one does not even know that for generic directions with respect to the Liouville measure, the boundary foliation is uniquely ergodic!);
- let μ be a measure of finite support on a cocompact Fuchsian group. Then we want to show that the harmonic measure for the random walk generated by μ is singular with respect to Lebesgue measure on the boundary of the hyperbolic plane.

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