An entropic tour of the Mandelbrot set

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Summary

1. Topological entropy

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2. External rays

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3. Main theorem, real version

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6. Remarks and conjectures

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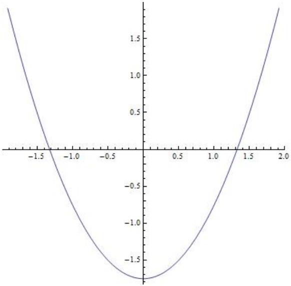

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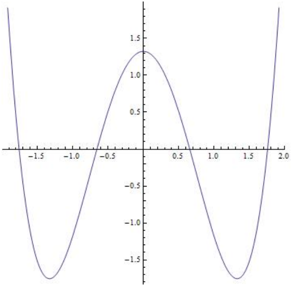

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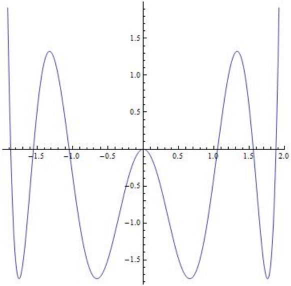

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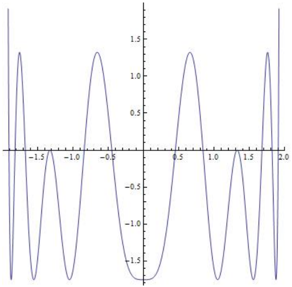

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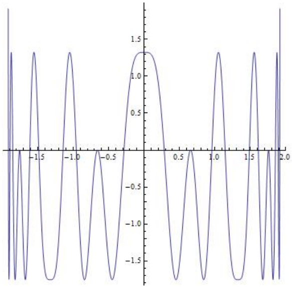

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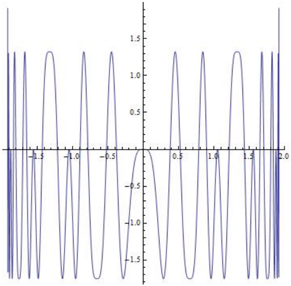

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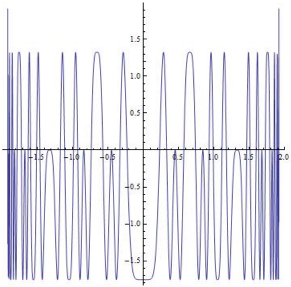

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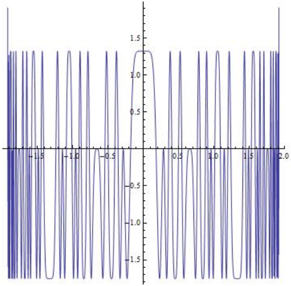

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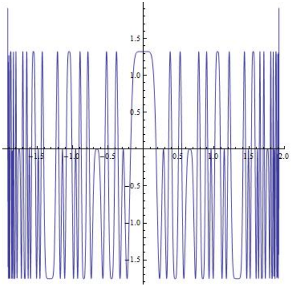
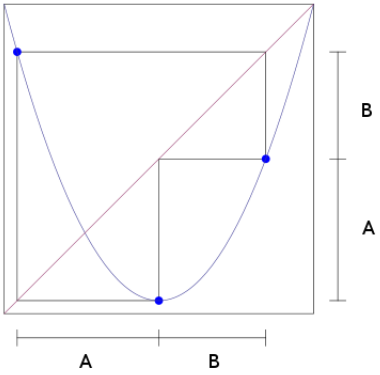


Entropy measures the randomness of the dynamics.

Example: the airplane map



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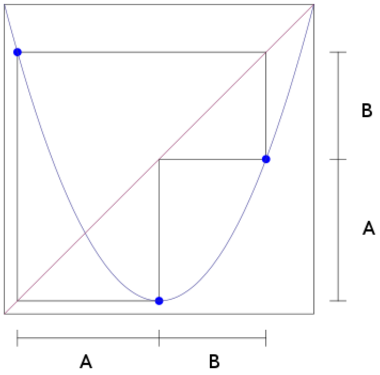

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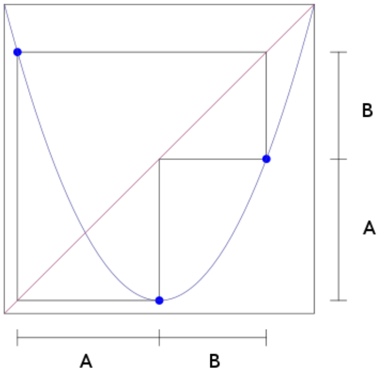

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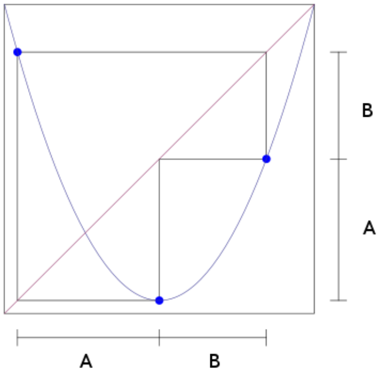

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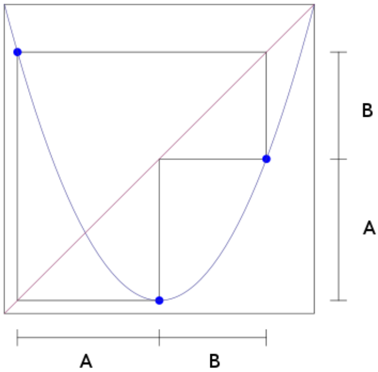

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How does entropy change with the parameter c?

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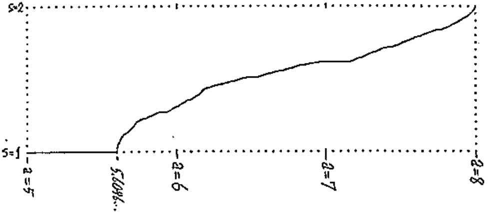
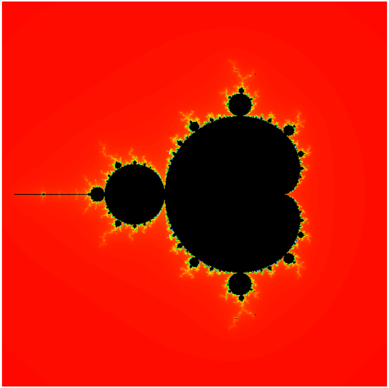


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$.

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \nrightarrow \infty\}$$



External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

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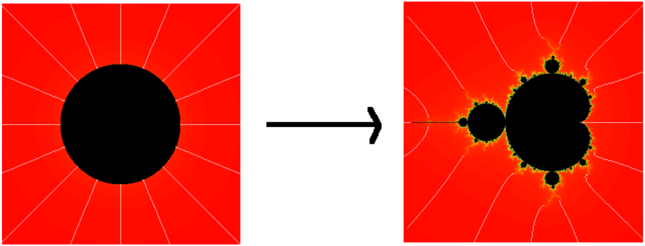

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The images of radial arcs in the disk are called external rays. Every angle $\theta \in S^1$ determines an external ray

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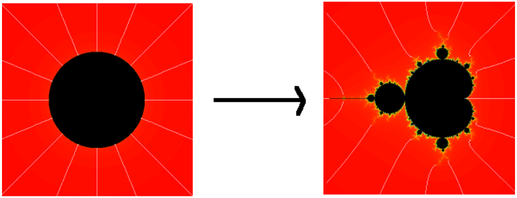
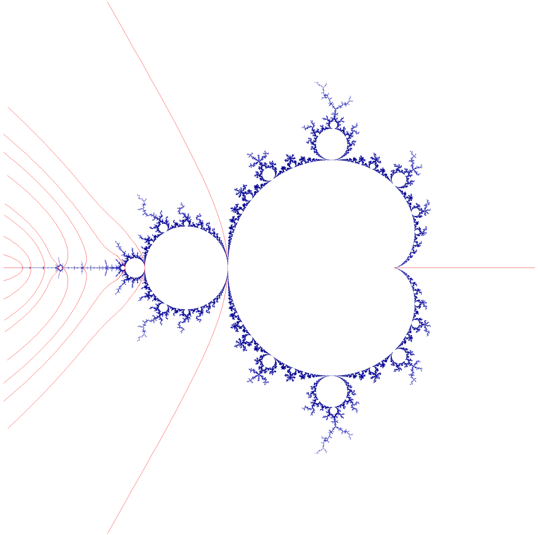


Rays landing on the real slice of the Mandelbrot set



Harmonic measure

Given a subset A of $\partial\mathcal{M}$, the harmonic measure $\nu_{\mathcal{M}}$ is the probability that a random ray lands on A:

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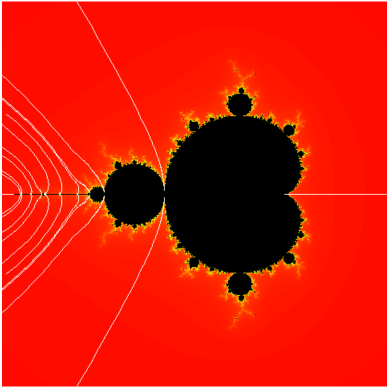

Real section of the Mandelbrot set

Theorem (Zakeri, 2000)

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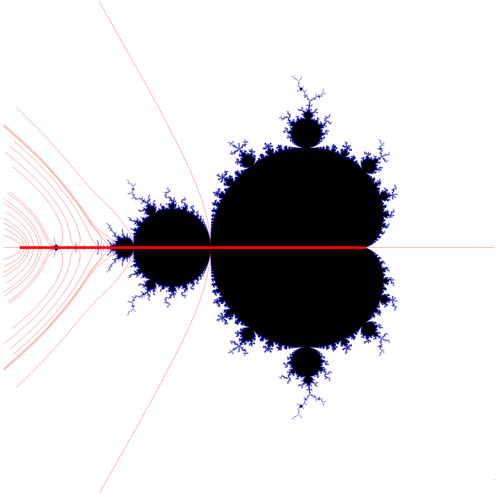

Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

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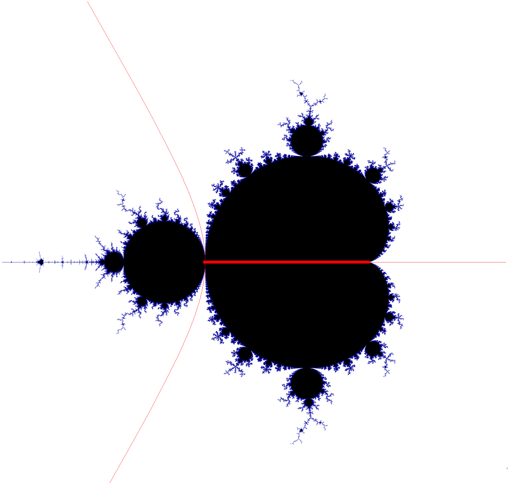

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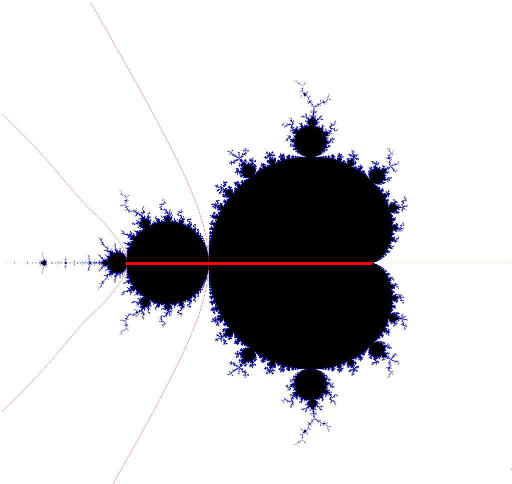

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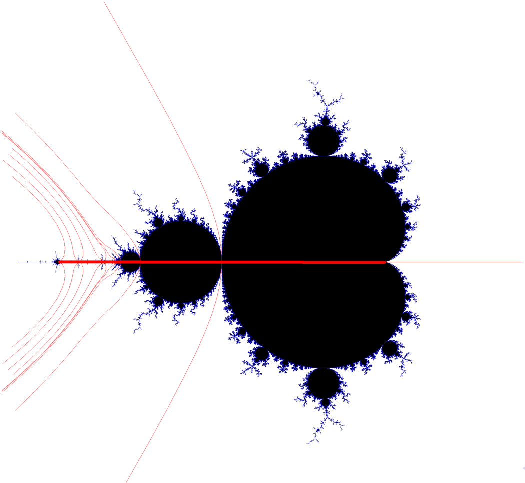

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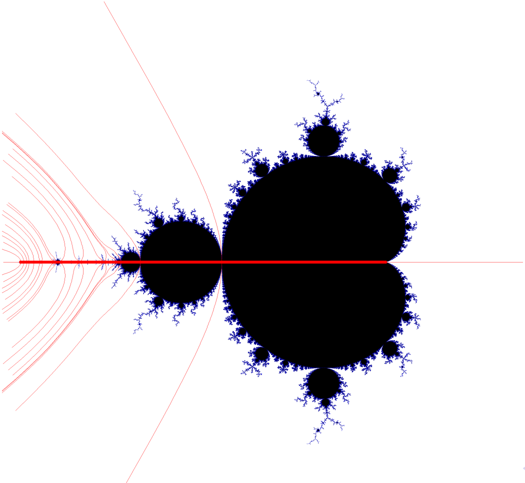

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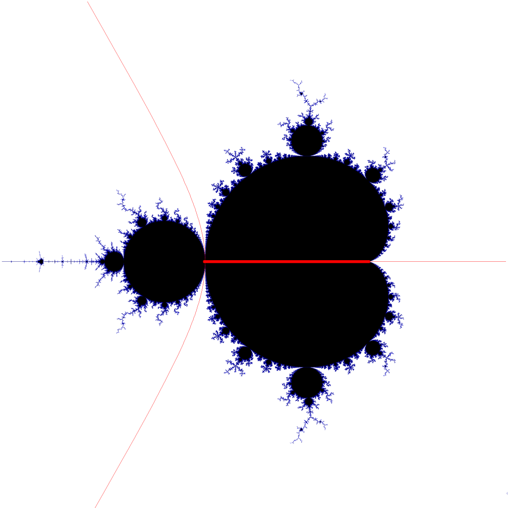
The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.



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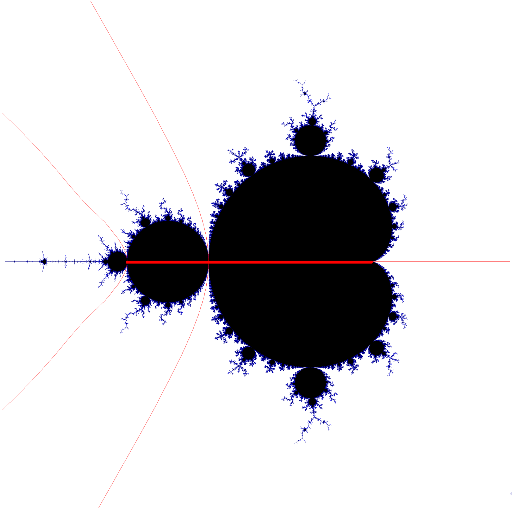

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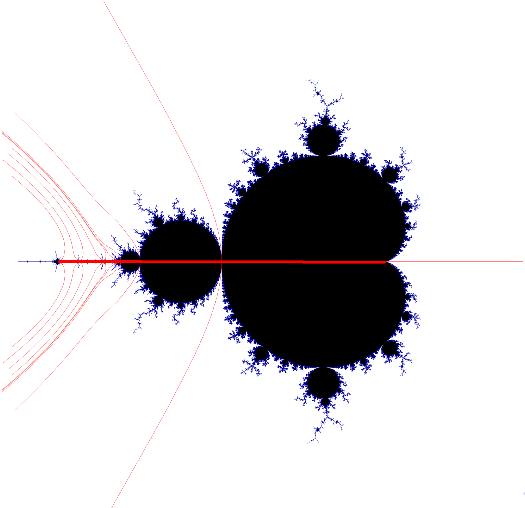

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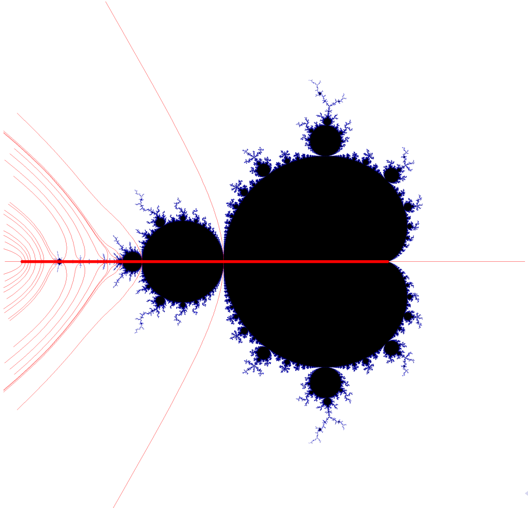

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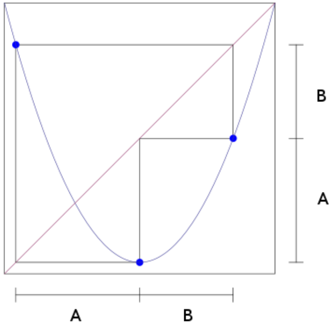

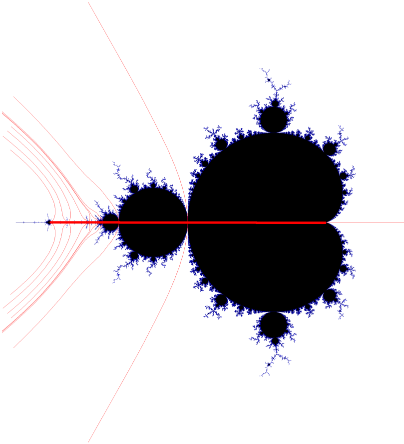

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- Entropy formula: relates dimension, entropy and Lyapunov exponent (Manning, Bowen, Ledrappier, Young, ...).
- The proof is purely combinatorial.
- It does not depend on MLC.
- It can be generalized to (some) non-real veins.

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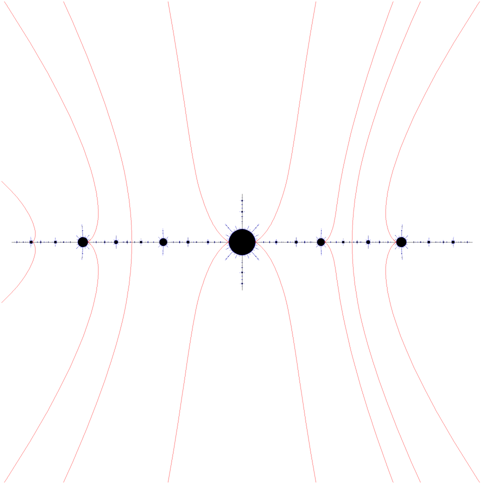

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Let

$$S_c := \{\theta \in S^1 \ : \ R(\theta) \text{ lands on } J_c \cap \mathbb{R}\}$$

the set of rays landing on the real section (spine) of the Julia set.

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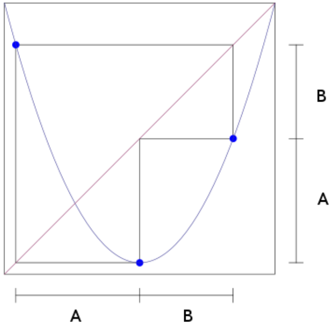

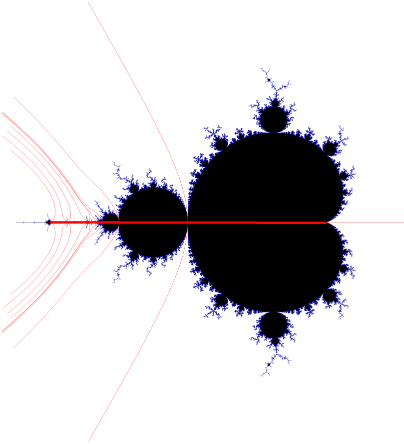

It also equals:

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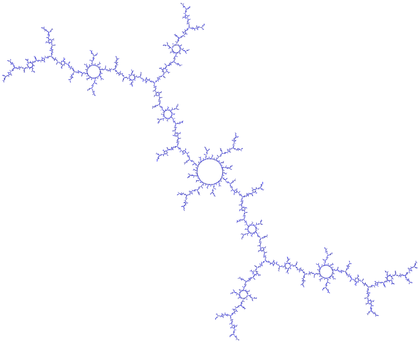
Corollary

The set of biaccessible angles for the Feigenbaum parameter (limit of period doubling cascades) c_{Feig} has Hausdorff dimension 0.

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is a forward invariant subset of the filled Julia set which contains the critical orbit.



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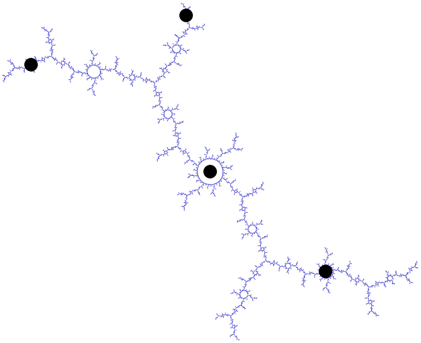

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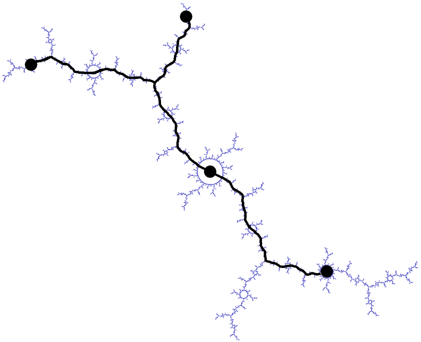

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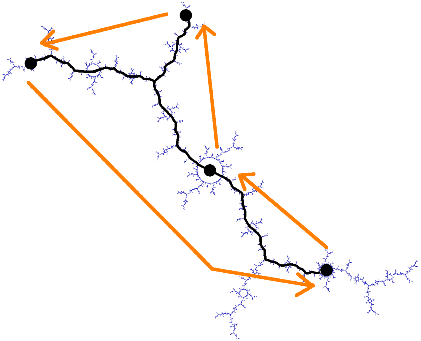

Topologically finite parameters

Definition

A parameter c is topologically finite if its Hubbard tree is homeomorphic to a finite tree.

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If c is biaccessible in parameter space (there are two distinct rays landing on c), then f_c is topologically finite (i.e. on all veins of $\mathcal{M}$).

Entropy of topologically finite parameters

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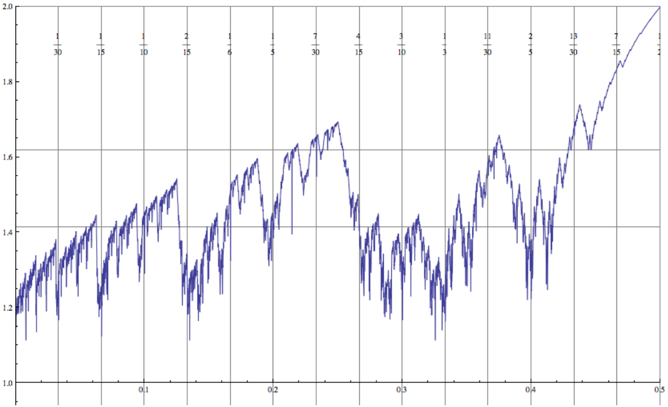
Entropy of Hubbard trees as a function of external angle (W. Thurston)



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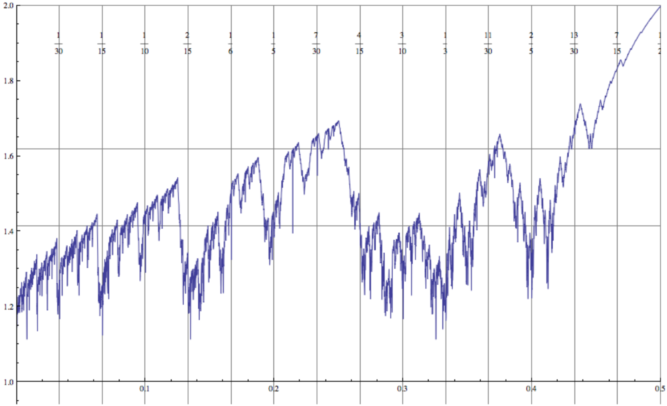
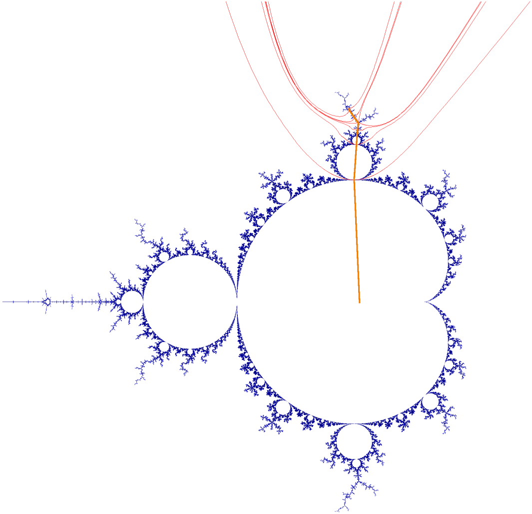


Can you see the Mandelbrot set in this picture?

The complex case

A vein is an embedded arc in the Mandelbrot set.



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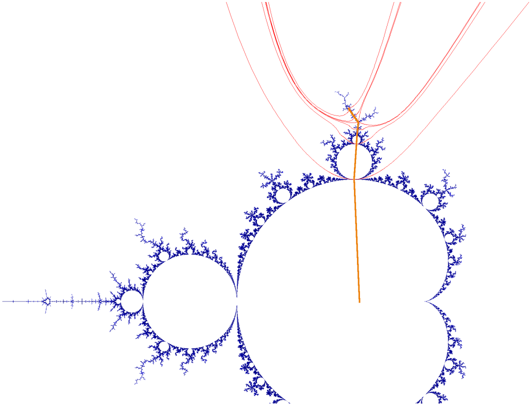

Given a parameter c along a vein, we can look at the set P_c of parameter rays which land on the vein "below" c.

Veins

Existence of veins converging to dyadic angles [Branner-Douady, Kahn, Riedl].

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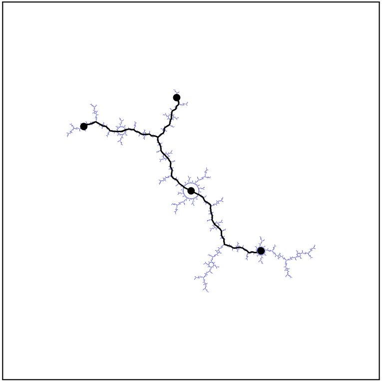
$$P_c := \{\theta \in S^1 \ : \ R_M(\theta) \text{ lands on } [0, c]\}$$

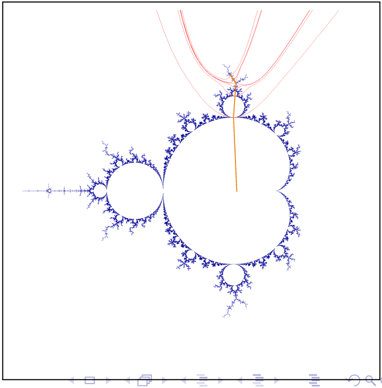
Complex version

Theorem

Let $v_{p/q}$ be the principal vein in the p/q-limb of the Mandelbrot set, and let $c \in v_{p/q}$. Then

$$\frac{h_{top}(f_c, T_c)}{\log 2} = \text{H.dim } H_c = \text{H.dim } P_c$$





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3. However:

Proposition

If c is a non-renormalizable, real parameter, and $c' > c$ another real parameter, there exists a non-constant, piecewise linear map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that

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6. By density of such parameters (in the space of angles), the result holds.

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E.g., the pseudocenter of the interval $[\frac{13}{15}, \frac{14}{15}]$ is $\frac{7}{8} = 0.111$, since $\frac{13}{15} = 0.\overline{1101}$ and $\frac{14}{15} = 0.\overline{1110}$.

Bonus level: a bisection algorithm

Theorem

Let $c_1 < c_2$ *be two real parameters on the boundary of* $\mathcal{M}$*, with external angles* $0 \leq \theta_2 < \theta_1 \leq \frac{1}{2}$.

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$$\theta^* = 0.s_1 s_2 \ldots s_{n-1} s_n$$

be its binary expansion, with $s_n = 1$.

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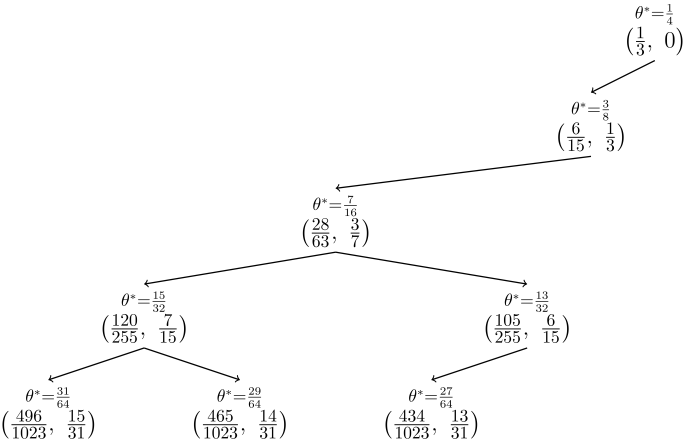
(where $\check{s}_i := 1 - s_i$). All hyperbolic windows are obtained by iteration of this algorithm, starting with $\theta_2 = 0$, $\theta_1 = 1/2$.

Bonus level: a bisection algorithm



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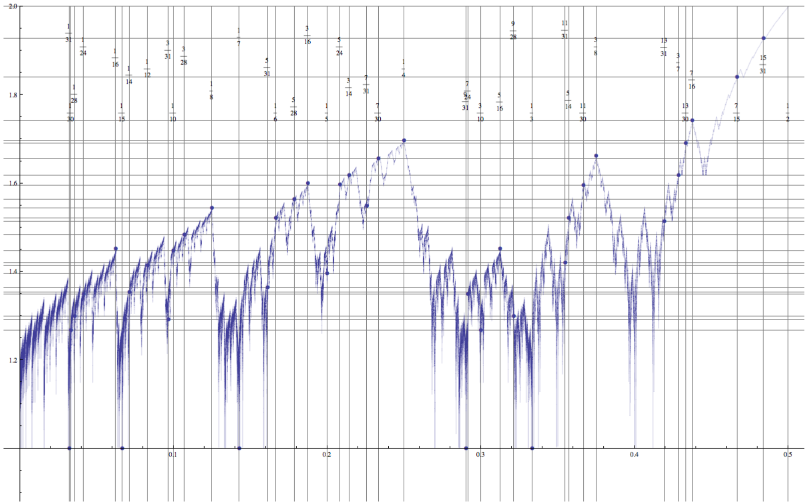
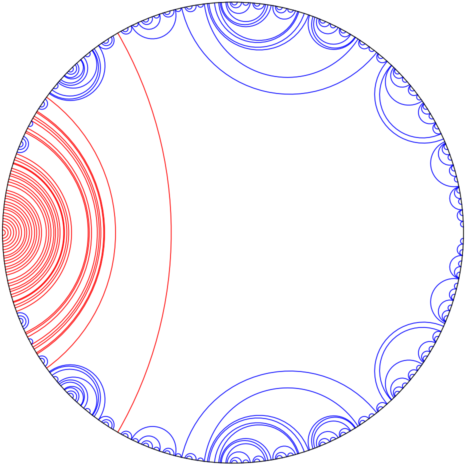
$$\max_{\theta \in [\theta_1, \theta_2]} h(\theta) = h(\theta^*)$$

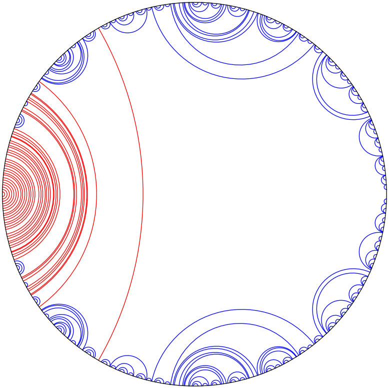
Thurston's entropy plot



Thurston's quadratic minor lamination



A transverse measure on QML



Let $\ell_1 < \ell_2$ two leaves, and τ a transverse arc connecting them.

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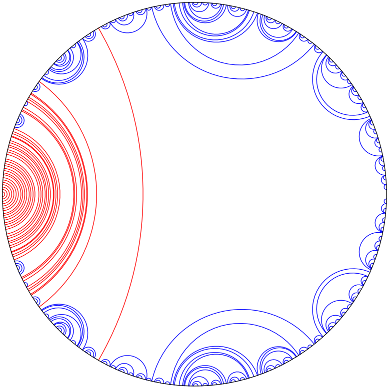

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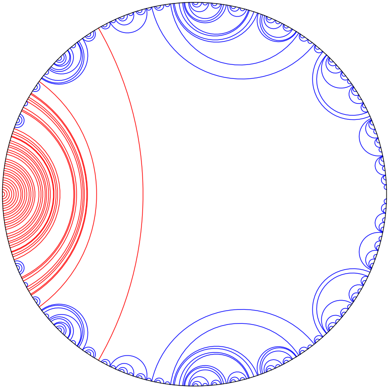

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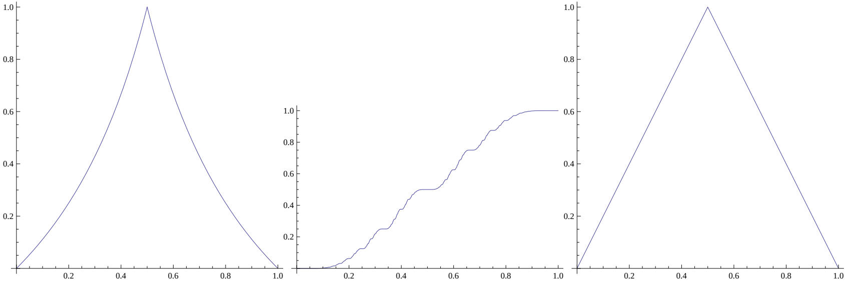
"Combinatorial bifurcation measure"?

The end

Thank you!

Coda: from Farey to the tent map, via ?

Minkowski's question-mark function conjugates the Farey map with the tent map



Continued fractions $\Leftrightarrow$ **Binary expansions**

The dictionary

Continued fractions	$\Leftrightarrow$	**Binary expansions**
$\mathcal{E}$	$\leftarrow ? \rightarrow$	$\mathcal{R}$
$\alpha -$ continued fractions		unimodal maps
numbers of generalized bounded type		external rays on Julia sets
cutting sequences for geodesics on torus (Cassaigne, 1999)		univoque numbers

A unified approach

The dictionary yields a unified proof of the following results:

1. The real part of the boundary of the Mandelbrot set has Hausdorff dimension 1

$$H.dim(\partial\mathcal{M} \cap \mathbb{R}) = 1$$

   (Zakeri, 2000)
2. The set of matching intervals for α-continued fractions has zero measure and full Hausdorff dimension (Nakada-Natsui conjecture, Carminati-T. 2010)
3. The set of univoque numbers has zero measure and full Hausdorff dimension (Erdős-Horváth-Joó, Daróczy-Kátai, Komornik-Loreti)

Entropy of α-continued fractions vs real hyperbolic components of $\mathcal{M}$