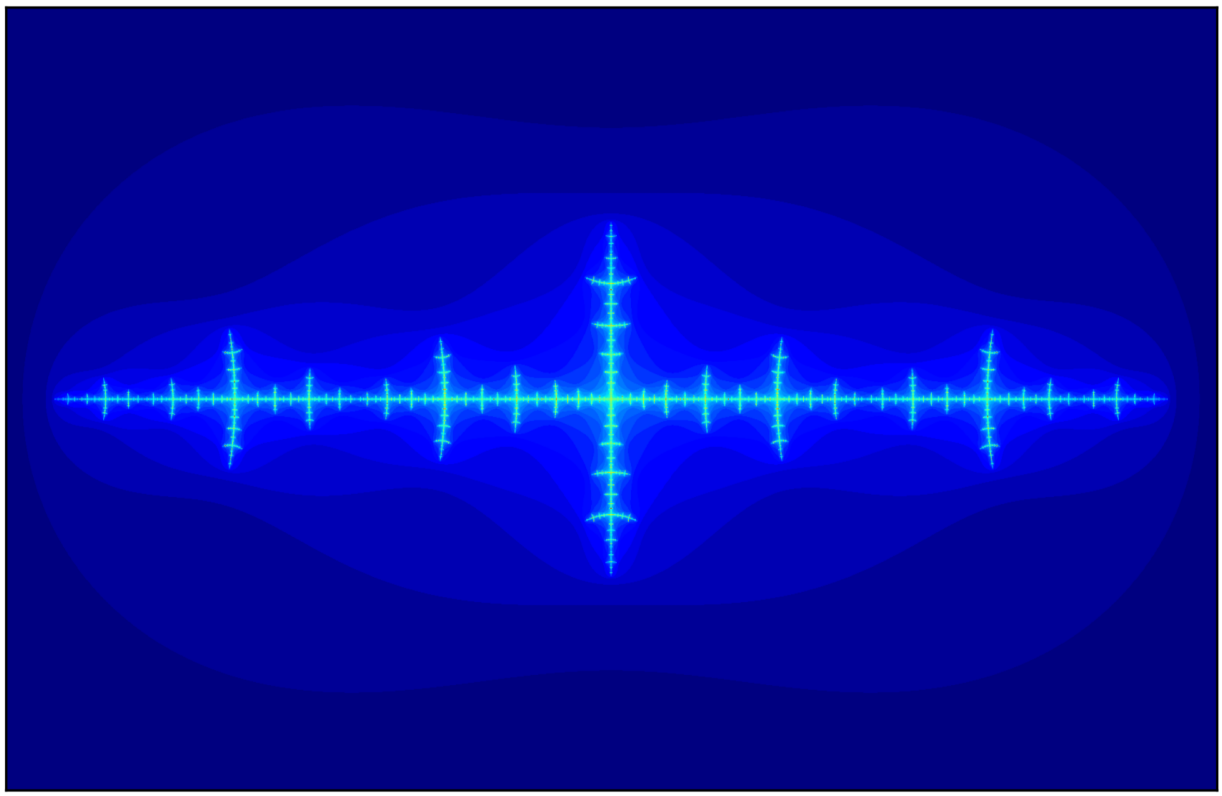
Similarity of the Mandelbrot Set and Julia Sets

A Closer Look at Misiurewicz Points



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Acknowledgements

I like to express my appreciation for the excellent article by Tan lei, *Similarity between the Mandelbrot set and Julia sets* (1990), on which this report is mainly based.

Moreover, I would like to thank professor Giulio Tiozzo for the interesting lectures on the topic of Holomorphic Dynamics, this past semester. Due to the Coronavirus, I have not been able to physically come to Toronto. However, thanks to him and my fellow students, I felt like I was there just a little. So, thank you.

I would also like to thank Lucas Levrouw, with whom I started this exciting exchange adventure. We supported each other along the way and got into the University of Toronto together. My dear Lucas, it was a crazy ride getting here, and it has not turned out the way it was supposed to, but still, what a journey. Thank you for walking this path with me.

Lastly, I would like to mention that the numerical parts of this project were done by me in Python. Specifically, SymPy and mostly NumPy were used.

Contents

1 Introduction

The Mandelbrot set

$$M \coloneqq \{c \in \mathbb{C} \mid f_c^n(0) \not\to \infty\}, \qquad f_c(z) \coloneqq z^2 + c$$

has been around for some forty years. The exact date is a bit harder to pinpoint, since it is a topic of debate who discovered it, see (Horgan, 1990). Already in 1978, Robert Brooks and J. Peter Matelski made the first picture of the Mandelbrot set, see figure 2. Only two years later, Mandelbrot published his paper on the set, with much better images of it due to his access to IBM's infrastructure. However it may be, Mandelbrot was able to acquire the public eye and made the set, and mathematics along with it, quite well-known and appreciated. It is mostly due to this that later in 1984, Douady and Hubbard named the set *the Mandelbrot set*.

The reason the name stuck, was because the (French) paper *Étude dynamique des polynômes complexes* (1984) by Douady and Hubbard was the first truly mathematical treatment of the Mandelbrot set, with actual profound results. In these so-called *Orsay notes*, quite a lot of today's classical work saw the light of day. And it is here already, in the first mathematical paper treating the Mandelbrot set, that the topic of this report finds its roots. These notes were not only written by Douady and Hubbard; a few people were collaborating on this project and making contributions to the text. One of these people was Lei Tan, see figure 1. Among other chapters, she wrote the very last one of these notes.

Figure 1: Tan Lei in 2015[1].

Later, these Orsay notes got translated into English. However, the last chapter is not present there. Tan took a bit longer and published this chapter as a separate research article, named *Similarity between the Mandelbrot set and Julia sets*, see (Tan, 1990). This was one of her first published articles.

Diving into the Mandelbrot set M or a Julia set

$$J_c \coloneqq \overline{\{z \mid z \text{ is repelling periodic under } f_c\}},$$

you would probably believe there are a lot of possible patterns in there. A pattern that is actually there, was looked into by Tan in her paper. Loosely stated, if you were to zoom in on so-called Misiurewicz points, with the right zoom in each step, you would find that what you are seeing is changing less and less; the set converges to some limiting set. It is said that around such a point, these sets are asymptotically self-similar. Maybe even more striking, if you take such a c in M, then it is automatically in J_c as well, and if you zoom in M and J_c at that point, they will also look more and more similar to each other. It is these results that Tan quantified, formalised, and proved and that will be discussed in this report.

The structure will be as follows. To quantify that two sets (locally) look similar, the Hausdorff–Chabauty distance will be introduced in section 2.1 after which the notion of (asymptotic) similarity is introduced in section 2.2. Lastly, the main actors are introduced in section 2.3: the Misiurewicz points. Then, in section 3, the result concerning Julia sets is presented and proved. Examples are also given to clarify the actual meaning of the theorem. Lastly, the Mandelbrot set gets in the picture, and a more general result is presented without proof. Also here, illustrative examples are provided.

It may be clear already, the result obtained by Tan is quite remarkable and very geometrical in nature. Moreover, she attained perfect precision in her results, as can be seen in the examples that follow.

Unfortunately, Tan died five years ago, on April 1, 2016, as young as 53. With this report, we hope that we can show a small part of her great work in the field of complex dynamics and appreciate these remarkable results of hers.

[1] Source of image: `math.univ-angers.fr/~tanlei/`

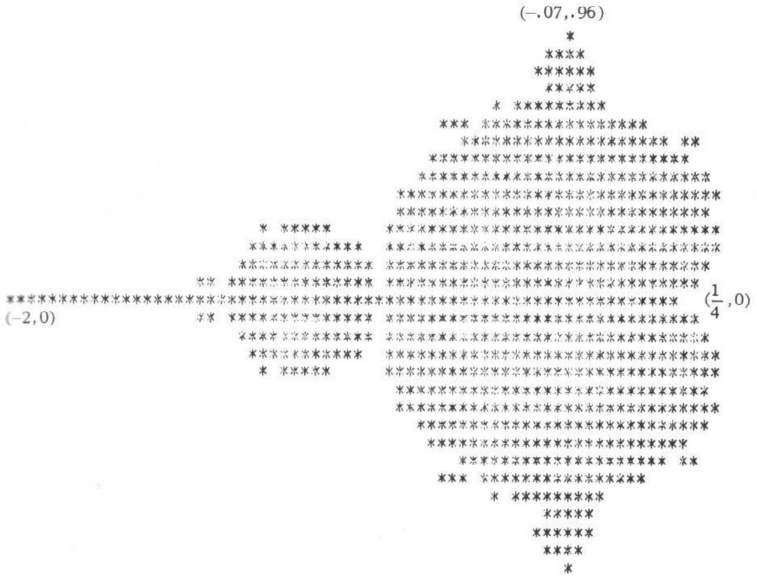

Figure 2: The first picture of the Mandelbrot set, see (Brooks and Matelski, 1978)

2 Formalisation

2.1 Hausdorff–Chabauty Distance

In this section, it is made precise what is meant by saying that a set converges to some kind of limiting set. To do so, we need a measure of how "far" two sets are from each other. Thinking about convergence, we need something that can express that a sequence of sets A_n comes closer and closer to a set B. Or, equivalently, that we can take smaller and smaller neighbourhoods of B in which the A_n keep getting into. The neighbourhoods we are looking for are given by the following definition.

Definition 1. Let $A \in \mathcal{F} \coloneqq \{S \subset \mathbb{C} \mid \emptyset \neq S \text{ compact}\}$ and $r \geq 0$. The r-hull of A, denoted by $H_r(A)$, is given by

$$H_r(A) \coloneqq \{z \in \mathbb{C} \mid d_E(z, A) \leq r\}, \tag{1}$$

where d_E is the euclidean distance.

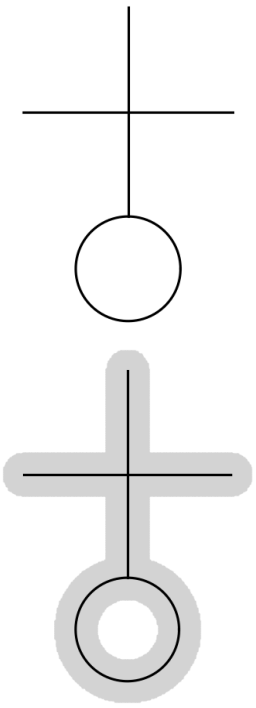

Figure 3: A set A and its r-hull $H_r(A)$ for some r.

In figure 3, we see an example of a r-hull; this is indeed the kind of neighbourhood we wanted. However, we need something more. We need to say that two sets A and B are close to each other, while now we can only say that A is close to B when A is in the r-hull of B for small r. Hence, we symmetrize this notion and then the definition of the Hausdorff distance is quite natural.

Definition 2 (Hausdorff Distance)**.** For $A, B \in \mathcal{F}$, we call

$$d(A, B) = \inf\{r \mid B \subset H_r(A) \text{ and } A \subset H_r(B)\} \tag{2}$$

the Hausdorff distance of A and B.

While it is not trivial to see with this definition, the Hausdorff distance makes $(\mathcal{F}, d)$ into a complete metric space.

Now we only have to do one last thing. Since we want to keep zooming in on a set and find that that converges to a limiting set, we want to discard parts of the set: we only need small parts of it. For example, suppose $0 \in A$. If we want to zoom in at A at 0, we take a little window around 0 and

Figure 4: A set A with the boundary of a disk $\mathbb{D}_r$ (left) and its r-window (right).

forget everything that is out of that window. Then we can zoom in a bit, and some information goes out of the scope of the window again, which we thus discard. We can take this window to be the closed disk $\overline{\mathbb{D}_r}$ of radius r to perhaps define $A \cap \overline{\mathbb{D}_r}$ to be the window around 0 with radius r. And indeed, this is almost exactly what we want. However, for technical reasons that we will not discuss here, this needs a little modification.

Definition 3. For a set A with $0 \in A$ and $r > 0$, the r-window of A is defined by

$$(A)_r = \left(A \cap \overline{\mathbb{D}_r}\right) \cup \partial\mathbb{D}_r, \tag{3}$$

where $\partial\mathbb{D}_r$ denotes the boundary of $\mathbb{D}_r$.

An example of this can be found in figure 4.

Remark 1. Of course, we do not only want to zoom in at zero. Therefore, we take the translation map τ_a defined by

$$\tau_a : \mathbb{C} \to \mathbb{C} : z \mapsto z + a. \tag{4}$$

Then, the r-window of A at a would be defined by

$$(\tau_{-a}A)_r. \tag{5}$$

With the notion of an r-window, we can now define the distance that we will be concerned with the most. It is just the Hausdorff distance with a restriction parameter.

Definition 4 (Hausdorff–Chabauty Distance)**.** For $A, B \in \mathcal{F}$ and $r > 0$, the Hausdorff–Chabauty distance is defined by

$$d_r(A, B) = d\left((A)_r, (B)_r\right). \tag{6}$$

2.2 Similarity: Definitions and Examples

Now that we can quantify how far two small parts of two images are, we are ready to think about similarity. The advantage of working with $\mathbb{C}$ instead of just the plane is the field structure that is present. Indeed, we know that multiplying the closed set A with $\rho = |\rho|e^{i\theta}$ is just performing a homothety of A with center 0 and ratio $|\rho|$ composed with a counter-clockwise rotation of A with θ radians. Hence, zooming in at $0 \in A$ (with which we mean the usual zooming ánd rotating) is encoded in the multiplication with the complex number ρ. Also note that we zoom in if $|\rho| > 1$ and we zoom out if $|\rho| < 1$. This leads to the following definition.

Definition 5. Assume $\rho = |\rho|e^{i\theta}$, with $|\rho| > 1$ and $\theta \in [0, 2\pi)$

- A closed subset $A \subset \mathbb{C}$ is **self-similar** about 0 with scale ρ if there is $r > 0$ such that

$$d_r(\rho A, A) = 0. \tag{7}$$

- A closed subset $A \subset \mathbb{C}$ is **asymptotically self-similar** about 0 with scale ρ if there exists $r > 0$ and a closed $B \subset \mathbb{C}$ such that

$$d_r\left(\rho^n A, B\right) \to 0 \text{ as } n \to \infty. \tag{8}$$

  We call B the **limit model** of A at 0.

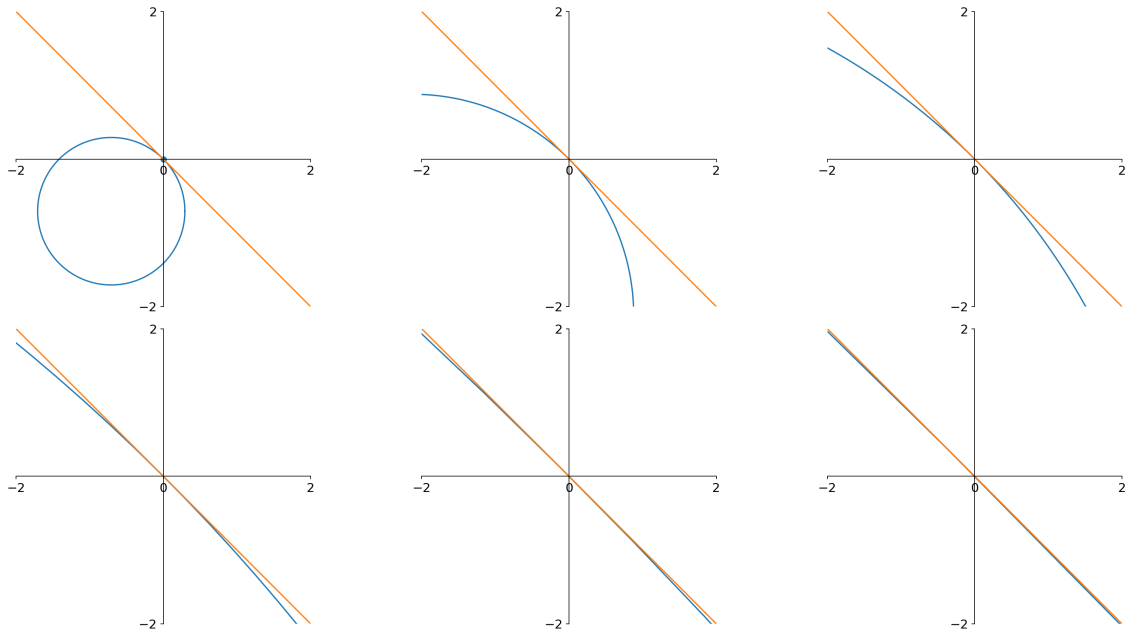

Figure 5: Illustration of asymptotic similarity about 0: multiplying a circle (and the tangent line) with 3^n for some low n. Note how the curve and tangent at 0 are looking more and more similar.

- Two closed sets $A, B \subset \mathbb{C}$ are **asymptotically similar** about 0 if there is $r > 0$ such that as for $\lambda \to \infty$ in $\mathbb{C}$,
$$d_r(\lambda A, \lambda B) \to 0, \tag{9}$$
or, equivalently, if
$$d_r(\rho^n A, \rho^n B) \to 0 \text{ as } n \to \infty. \tag{10}$$

Remark 2. Note that the equivalence stated in the last definition requires some (rather technical) work to prove. Intuitively though, this is quite clear. Since you multiply both A and B with the same factor λ, they are both zooming in and rotating in exactly the same way. Hence, A and B should come closer, irrelevant of the rotations, and irrelevant of the magnitude with which you zoom. Thus replacing $\lambda \to \infty$ with a specific way of going to infinity, like ρ^n, is justified.

We now give examples to clarify these key concepts.

Example 1. If you have a line $\ell = \{\mu z_0 \mid \mu \in \mathbb{R}\}$ in the plane for some $z_0 \neq 0$, we easily see that it is self-similar about 0 with as scale any $\rho \in \mathbb{R}_{>1}$. Indeed, for any $r > 0$, we have
$$d_r(\rho\ell, \ell) = d_r(\ell, \ell) = 0.$$

Example 2. If a curve is smooth at a point, the tangent and the curve are asymptotically similar at that point. This requires some (rather technical) work to prove, but is intuitively clear. See also figure 5.

2.3 Misiurewicz Points

To apply the ideas of the last sections to the Mandelbrot set M or to a Julia set J_c associated to $f_c(z) = z^2 + c$, we need one more concept. The parameters $c \in M$ that will be the protagonists of this report are called Misiurewicz points and they are defined as follows.

Definition 6. A point $c \in M$ is called a Misiurewicz point if 0 is eventually periodic, but not periodic. So, there are integers k, ℓ such that
$$f_c^k\left(f_c^\ell(c)\right) = f_c^\ell(c). \tag{11}$$
Here, ℓ is called the pre-period of c, and k is the period of $f_c^\ell(c)$.

Example 3 (Misiurewicz Points)**.** Two simple examples are given by $c = -2$ and $c = i$. Indeed

$$-2 \overset{f_{-2}}{\longmapsto} (-2)^2 - 2 = 2 \overset{f_{-2}}{\longmapsto} 2^2 - 2 = 2 \overset{f_{-2}}{\longmapsto} \dots, \qquad \text{so } k = 1, \ell = 1,$$

$$i \overset{f_i}{\longmapsto} -1 + i \overset{f_i}{\longmapsto} -i \overset{f_i}{\longmapsto} -1 + i \overset{f_i}{\longmapsto} \dots, \qquad \text{so } k = 2, \ell = 1.$$

We have the following two classical results about Misiurewicz points.

Proposition 1. The set of Misiurewicz points is a countable and dense subset of the boundary of M.

Proposition 2 (Douady and Hubbard)**.** If c is a Misiurewicz point, then 0 (and thus c) is eventually repelling periodic.

These results tell us that firstly, there are quite a lot of Misiurewicz points and that secondly, a $c \in M$ that is Misiurewicz has the property that $c \in J_c$.

3 The Julia Set

3.1 Theorem Statement

In this section, we will prove that the Julia set J_c is asymptotically self-similar about a Misiurewicz point c. Put more precisely, we will show the following result.

Theorem 1 (Tan)**.** Assume c is Misiurewicz. Let $\ell, k \in \mathbb{Z}$ be minimal such that

$$f_c^k \left(f_c^\ell(c) \right) = f_c^\ell(c).$$

Let $\alpha = f_c^\ell(c)$ and $\rho = (f_c^k)'(\alpha)$. Then

- $|\rho| > 1$, $(f_c^\ell)'(c) \neq 0$
- There is a conformal mapping φ defined in a neighbourhood of α, with $\varphi(\alpha) = 0$, $\varphi'(\alpha) = 1$ and $\varphi \circ f_c^k \circ \varphi^{-1}(z) = \rho z$.
- There are $r > 0$, a neighbourhood V of α and U of c such that $f_c^\ell(\overline{U}) = \overline{V}$ and as $n \to \infty$,

$$d_r \left(\rho^n(\tau_{-c} J_c), \frac{1}{(f_c^\ell)'(c)} \varphi \circ f_c^\ell(J_c \cap \overline{U}) \right) \to 0.$$

This will be a simple corollary of a theorem that we now set out to prove.

3.2 Preliminary Results

We list three lemma's that will be used in the proof of the theorem. The first is a simple set-theoretical result.

Lemma 1. Suppose A is a completely invariant subset of f, i.e. $f(A) = f^{-1}(A) = A$, then for all $U \subset \mathbb{C}$ it holds that

$$f(A \cap U) = A \cap f(U). \tag{12}$$

Consequently,

$$f^{-1}(A \cap f(U)) \cap U = A \cap U. \tag{13}$$

We also recall Koenigs's linearisation theorem.

Theorem 2 (Koenigs)**.** Suppose U, V are neighbourhoods of x in $\mathbb{C}$ and $g : U \to V$ is a holomorphic function with $g(x) = x$ and $|g'(x)| \neq 0, 1$. If $\rho = g'(x)$, then there is a conformal mapping φ defined in a neighbourhood of x with $\varphi(x) = 0$, $\varphi'(x) = 1$ such that

$$\varphi \circ g \circ \varphi^{-1}(z) = \rho z.$$

Moreover, φ is explicitly given by

$$\varphi(z) = \lim_{n\to\infty} \rho^n \left((g^{-1})^n(z) - x\right),$$

where g^{-1} is the unique inverse of g in a neighbourhood of x.

Lastly, we remember that we saw that a smooth curve and a tangent were asymptotically similar at the tangent point. This is easily generalised to mappings between subsets of $\mathbb{C}$. Moreover, if the image $f(A)$ of A under a conformal mapping is asymptotically self-similar, then we expect that this structure was transported under f from A itself. This is exactly the content of the next result.

Proposition 3. Suppose that U, V are two neighbourhoods of 0 in $\mathbb{C}$ and let $f : U \to V$ be a conformal map that keeps 0 fixed. If $A \subset U$ is closed such that $f(A)$ is asymptotically self-similar about 0 with scale ρ and limit model B, then A is asymptotically self-similar about 0, with the same scale and limit model $\frac{1}{f'(0)}B$.

Proof. We have that $f'(0)A$ is asymptotically similar to $f(A)$ about 0, and by assumption, $f(A)$ is asymptotically self-similar about 0 with scale ρ and limit model B. Hence, we can find $r > 0$ such that as $n \to \infty$,

$$d_r(\rho^n f(A), B) \to 0 \qquad\qquad d_r(\rho^n f(A), \rho^n f'(0)A) \to 0.$$

Applying the triangle inequality, we see

$$d_r(f'(0)\rho^n A, B) \leq d_r(\rho^n f'(0)A, \rho^n f(A)) + d_r(\rho^n f(A), B) \to 0.$$

Since f is conformal, we have $f'(0) \neq 0$, whence we can set $s = r/|f'(0)|$ to obtain

$$d_s\left(\rho^n A, \frac{1}{f'(0)}B\right) \to 0. \qquad \square$$

With these results, we can now start with the real work.

3.3 Proof of theorem 1

The theorem that will imply theorem 1 is now stated.

Theorem 3. Let f be a rational map and A be a completely invariant closed set under f. Suppose that x is an eventually repelling periodic point for f, so that

$$f^k(f^\ell(x)) = f^\ell(x).$$

Then there exists $r > 0$, and a conformal mapping φ defined in a closed neighbourhood $\overline{U}$ of x, such that

$$d_r\left(\rho^n \tau_{-x} A, \frac{1}{\varphi'(x)}\varphi(A \cap \overline{U})\right) \to 0, \quad \text{where } \rho = (f^k)'(f^\ell(x)).$$

So, A is asymptotically self-similar about x with scale ρ and limit model $\frac{1}{\varphi'(x)}\varphi(A \cap \overline{U})$. Moreover,

- if $\ell = 0$: we can choose φ such that $\varphi'(x) = 1$.
- if $\ell > 0$ and $(f^\ell)'(x) \neq 0$: then the limit models of A at x and at $f^\ell(x)$ are the same, up to multiplication by $(f^\ell)'(x)$.

We only prove this in the case that x is periodic, i.e. $\ell = 0$. The main ideas are then addressed, and when there are no critical points in the orbit of x, you can immediately apply the periodic case. If there are critical points, however, it becomes a bit trickier and more technical. Luckily, this is also the case of this result that we do not need.

Proof in case x is periodic. Suppose x is periodic.

Note that x is a fixed point for f^k. Also, since x is assumed repelling, we have $|\rho| > 1$. We can thus apply Koenigs's theorem to find neighbourhoods V_x and V_0 of x and 0 respectively, and a conformal map $\varphi : V_x \to V_0$ such that $\varphi(x) = 0$, $\varphi'(x) = 1$ and

$$\varphi \circ f^k = \rho \cdot \varphi.$$

Now take $r > 0$ such that $\overline{\mathbb{D}_r} \subset V_0$ and let s be such that $sr = |\rho|$. We define a neighbourhood $U = \varphi^{-1}(\mathbb{D}_s)$ of x and let $B = \varphi(A \cap \overline{U})$. By proposition 3, we only have to show that B is asymptotically self-similar with scale ρ to find that A (by further restriction of $A \cap \overline{U}$) is asymptotically self-similar with scale ρ. We do this by showing that B is actually self-similar with scale ρ.

Since A is completely invariant under f, it is also completely invariant under f^k. Playing a bit with the information at hand and using lemma 1, we then find that

$$f^k(A \cap U) \cap U = A \cap U$$

or that,

$$f^k((A \cap \overline{U}) \cap U) \cap U = (A \cap \overline{U}) \cap U.$$

We apply φ to this and use injectivity of φ, as well as the conjugation of f^k:

$$\rho\varphi((A \cap \overline{U}) \cap U) \cap \varphi(U) = \varphi(A \cap \overline{U}) \cap \varphi(U).$$

Per definition of U, we then find

$$\rho\left[\varphi(A \cap \overline{U}) \cap \mathbb{D}_s\right] \cap \mathbb{D}_s = \varphi(A \cap \overline{U}) \cap \mathbb{D}_s,$$

so that

$$\rho\left[B \cap \mathbb{D}_s\right] \cap \mathbb{D}_s = B \cap \mathbb{D}_s.$$

We then move ρ inside and use that $s < r$ to find

$$\rho B \cap \mathbb{D}_r \cap \mathbb{D}_s = \rho B \cap \mathbb{D}_s = B \cap \mathbb{D}_s.$$

By taking a neighbourhood of 0 which is a bit smaller, we can then take the closure of the disk

$$\rho B \cap \overline{\mathbb{D}_{s/2}} = B \cap \overline{\mathbb{D}_{s/2}},$$

from which it follows that

$$(\rho B)_{s/2} = (B)_{s/2}.$$

That is, B is self-similar with scale ρ. This means that B is certainly asymptotically self-similar with scale ρ and limit model B.

Now applying proposition 3, we find what we set out to prove. □

Proof of theorem 1. Since the Julia set J_c is a completely invariant subset of f_c and for c Misiurewicz, c is eventually repelling periodic, the result follows from theorem 3. The only thing we need to show is that $(f_c^\ell)'(c) \neq 0$. This we see by noting that 0 is the only critical point of f_c, so that

$$(f_c^\ell)'(c) = 2^\ell \cdot f_c^{\ell-1}(c) \cdot f_c^{\ell-2}(c) \cdot \ldots \cdot f_c(c) \cdot c \neq 0,$$

since 0 is not periodic under f_c for c Misiurewicz. □

3.4 Explorations and Examples

We now give examples to illustrate theorem 1.

As a first example, we take $c = i$ to show the machinery at work[2]. We already saw that $k = 2, \ell = 1$ for the minimal k, ℓ such that $f_c^k\left(f_c^\ell(c)\right) = f_c^\ell(c)$. We can then calculate the multiplier ρ to find

$$\rho = (f_c^k)'(f_c^\ell(c)) = [(z^2+c)^2+c]'\Big|_{z=c^2+c=i-1} = 4(1+i) \approx 5.66e^{i\pi/4}.$$

Hence, we have to zoom with factors $5.66, 5.66^2, 5.66^3, \ldots$ and rotate with $45°$ each time to converge to the limiting set. The results can be found in figure 6 on page 9. In these figures, the (rescaled) number of iterations needed to diverge are plotted, with a cut-off of 400 iterations (when this cut-off is reached, the point under consideration is assumed to be in the Mandelbrot set).

As another example, we take $c \approx -0.6368 - 0.685i$. We repeat the process and find $k = 1, \ell = 4$ and $\rho \approx 3.08e^{i\theta}$, $\theta \approx 12.82°$. The results are found in figure 7 on page 10.

4 Julia and Mandelbrot

4.1 Statement

The previous section shows that if we zoom in on the Julia set J_c at c for c a Misiurewicz point, we get closer and closer to a limit model, in the sense of the Hausdorff–Chabauty distance. Moreover, this limit model was in theory locally achievable (via Koenigs's theorem), and not only by zooming in indefinitely. This result is quite nice and remarkable, and yet, it can be better. A Misiurewicz point is a point of the boundary of M, and so we might expect a fractal structure in M around c as well. The following theorem asserts that this fractal structure around a Misiurewicz point is similar to that of the Julia set J_c.

Theorem 4. Let c be a Misiurewicz point. There are ρ with $|\rho| > 1$, $r, r', s > 0$, a closed set $L \subset \mathbb{C}$ with $\rho L = L$, and $\lambda \in \mathbb{C}_{\neq 0}$ such that as $n \to \infty$,

$$\begin{aligned} d_r(\rho^n \tau_{-c} J_c, L) &\to 0 \\ d_{r'}(\rho^n \tau_{-c} M, \lambda L) &\to 0 \\ d_s(\rho^n \tau_{-c} M, \rho^n \lambda \tau_{-c} J_c) &\to 0. \end{aligned}$$

Remark 3. Note that, from the previous section, we know what ρ and L are. Left to say what λ is. Since $\alpha \coloneqq f_c^\ell(c)$ is a fixed point of f_c^k, we can apply the implicit function theorem to find a neigbourhood U of c and a mapping $\widetilde{\alpha} : U \to \mathbb{C}$ such that $\widetilde{\alpha}(c) = \alpha$ and such that $f_c^k(\widetilde{\alpha}(c)) = \widetilde{\alpha}(c)$. Tan then proves that

$$\lambda = \frac{(f_c^\ell)'(c)}{\frac{\mathrm{d} f_\kappa^\ell(\kappa)}{\mathrm{d}\kappa}\Big|_{\kappa=c} - \frac{\mathrm{d}\widetilde{\alpha}(\kappa)}{\mathrm{d}\kappa}\Big|_{\kappa=c}}.$$

Thus, this theorem tells us that about a Misiurewicz point, the Mandelbrot set is also asymptotically self-similar, and with the same scale as J_c. Moreover, their limit models are related by one complex parameter λ. Taking this parameter into account, we thus also find that M and J_c are asymptotically similar about c.

The proof of this theorem is quite involved and consists mostly of checking the conditions to apply a modified version of another proposition. It would bring us too far to discuss the proof here, but we believe that the key ideas have been discussed in the course of this report. We now give examples that show this theorem in action.

[2]This Misiurewicz point is an obvious point to do this, as it has not too many decimals.

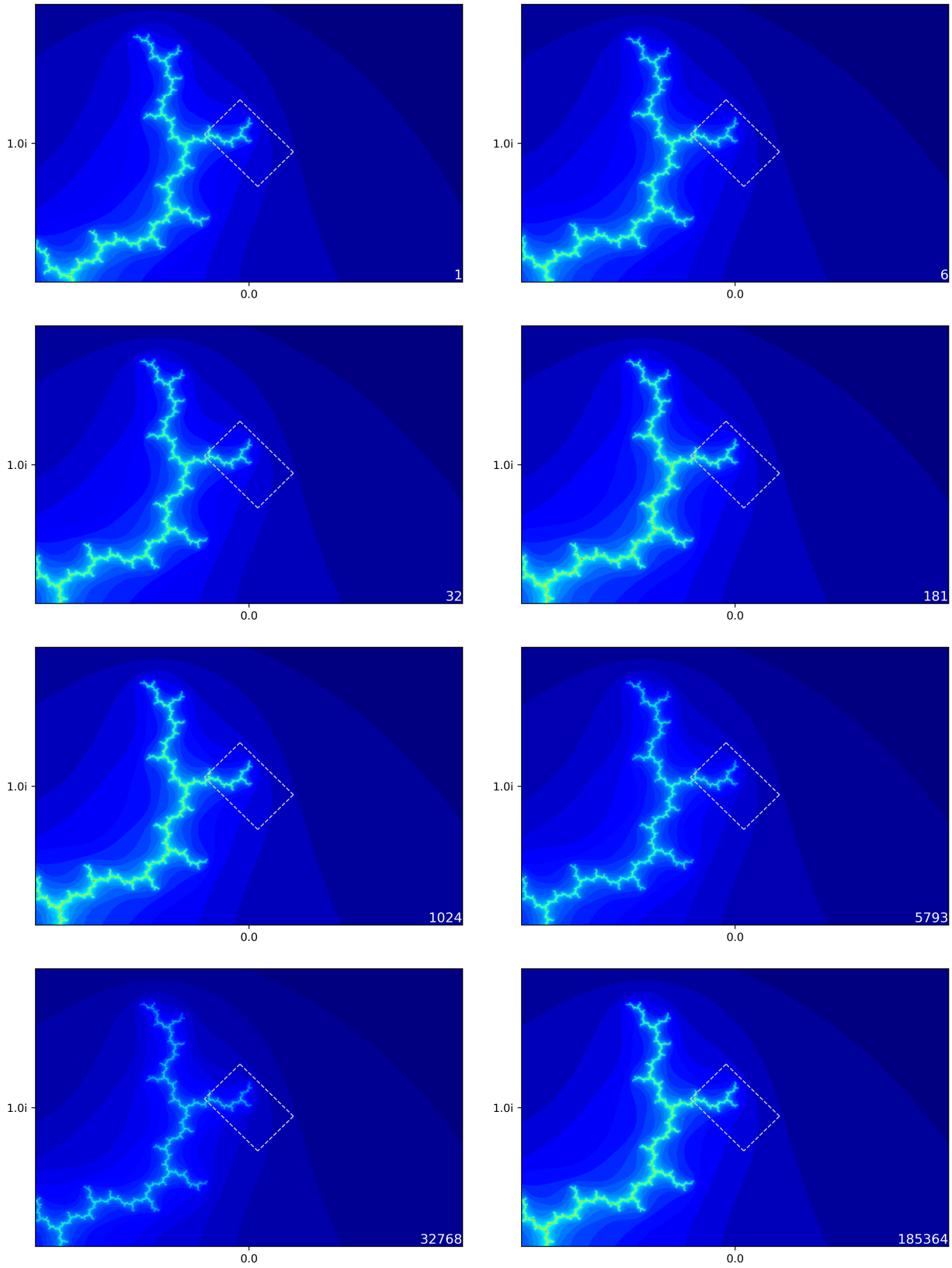

Figure 6: Zooming in on the Julia set J_i at the point i. The zooming factor is 5.66^n starting from the picture in the upper left corner. This (rounded) factor is written in each image. The rectangle in each image shows the part of the image that is detailed more in the next image. The first image shows the region $[-0.3, 0.3] \times [1 - 0.2, 1 + 0.2]i \subset \mathbb{C}$.

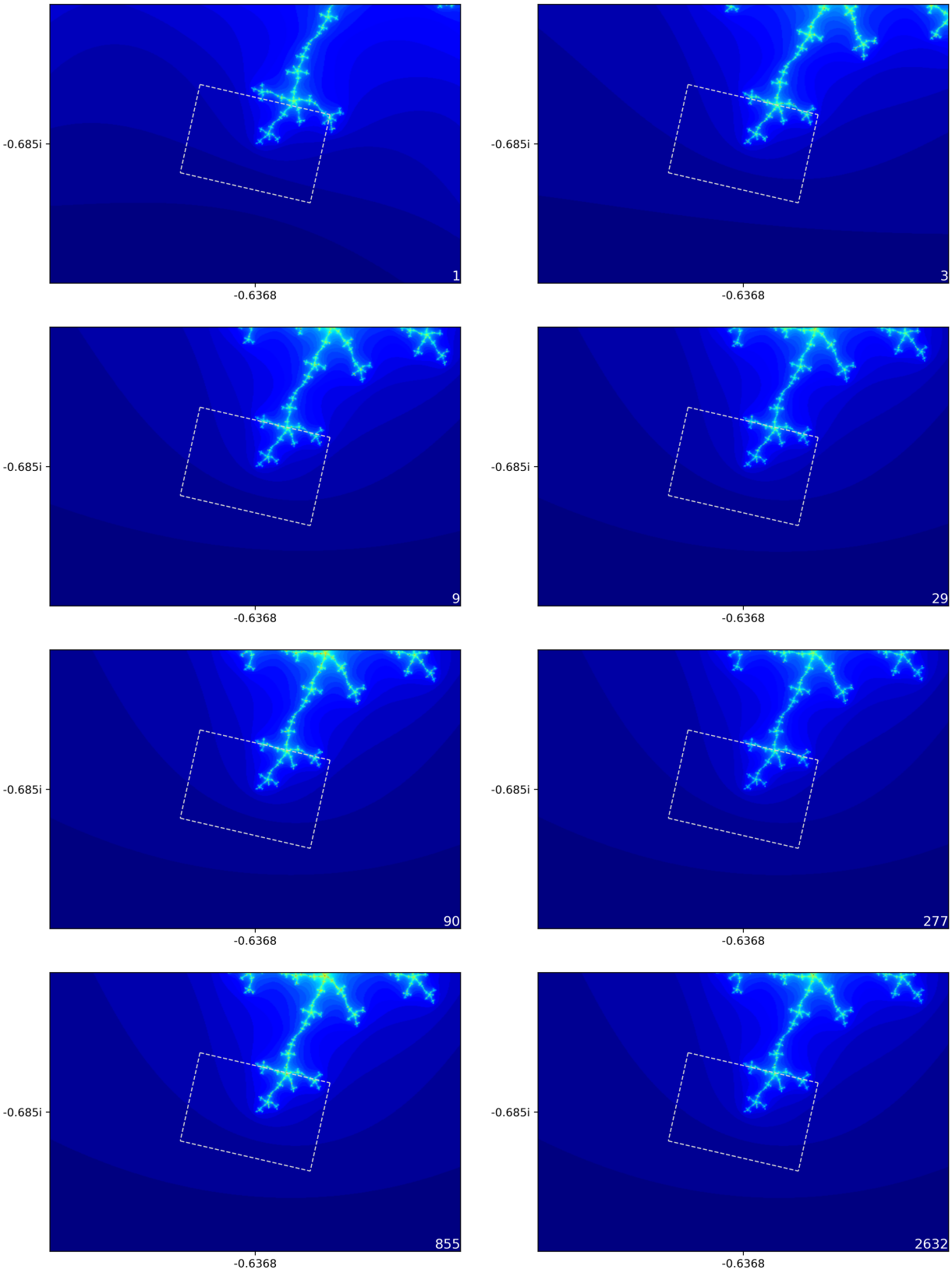

Figure 7: Zooming in on the Julia set J_c at the point $c \approx -0.6368 - 0.685i$. The zooming factor is 3.08^n starting from the picture in the upper left corner. This (rounded) factor is written in each image. The rectangle in each image shows the part of the image that is detailed more in the next image. The first image shows the region $[-0.6368 - 0.3, -0.6368 + 0.3] \times [-0.685 - 0.2, -0.685 + 0.2]i \subset \mathbb{C}$.

4.2 Examples

We consider again the Misiurewicz point $c = i$. We calculate λ of theorem 4 as suggested by remark 3. We see with $\ell = 1$ that

$$(f_c^\ell)'(c) = f_c'(c) = 2c = 2i, \tag{14a}$$

$$\left.\frac{\mathrm{d}f_\kappa^\ell(\kappa)}{\mathrm{d}\kappa}\right|_{\kappa=c} = \left.\frac{\mathrm{d}(\kappa^2+\kappa)}{\mathrm{d}\kappa}\right|_{\kappa=c} = 2c+1 = 2i+1. \tag{14b}$$

Now, observe that $\widetilde{\alpha}$ is a solution of $f_c^\ell(\widetilde{\alpha}(c)) = \widetilde{\alpha}(c)$ near $\alpha = f_c^\ell(c) = \widetilde{\alpha}(c) = i-1$. That is, $\widetilde{\alpha}(c)^2 + c = \widetilde{\alpha}(c)$. Implicitly differentiating this, we obtain

$$\left.\frac{\mathrm{d}\widetilde{\alpha}(\kappa)}{\mathrm{d}\kappa}\right|_{\kappa=c} = \frac{-1-2(\widetilde{\alpha}(c)^2+c)}{4\widetilde{\alpha}(c)\cdot[\widetilde{\alpha}(c)^2+c]-1} = \frac{2i-1}{3+4i}. \tag{15}$$

Hence, we find

$$\lambda = 1 + \frac{1}{2}i \approx 1.12e^{i\theta}, \qquad \theta \approx 26.6^\circ. \tag{16}$$

This, together with the information of the examples on page 8, gives us enough information to see the theorem in action. The results can be found in figure 8, page 12. You can compare the first column of this figure with figure 6.

We can also do the previous computation of λ for other Misiurewicz points. For $c \approx -0.6368 - 0.685i$, we find

$$\lambda \approx 0.9468 - 0.3006i \approx 0.9934e^{i\theta}, \qquad \theta \approx -17.6140^\circ. \tag{17}$$

The results for this point are found on page 13, figure 9. You can again compare to figure 7.

5 Conclusions

The phenomenon of (asymptotic) self-similarity of the Julia set and of the Mandelbrot set was quantified and made precise already four years after the first detailed image of the Mandelbrot set.

In this short timeframe, Tan Lei wrote down the results and proofs that were discussed in this paper. She showed that the Julia set of f_c and the Mandelbrot set are asymptotically self-similar about any Misiurewicz point c. As was noted, these Misiurewicz points were quite abundant in the boundary of the Mandelbrot set, which means that her results apply to a lot of parameters c.

In proving her results, she could write down the precise details on how to practically see these results in action. This was gratefully used here to experimentally discuss and verify her results, not in the least in showing the use of the parameter λ.

Another interesting consequence of her results is that when looking at a particular Julia set J_c, you can make an educated guess of your c. Indeed, you can search in the Mandelbrot set for a shape roughly similar to your Julia set to find an estimate of your c. Also compare to the first row of figures 8 and 9.

In conclusion, Tan lei made precise a geometric and intuitive phenomenon with extreme precision. In doing so, she made clear another intricate and interesting connection between the Mandelbrot set and Julia sets of quadratic polynomials. Moreover, she did this in an extremely short timeframe, which is impressive in its own right already.

References

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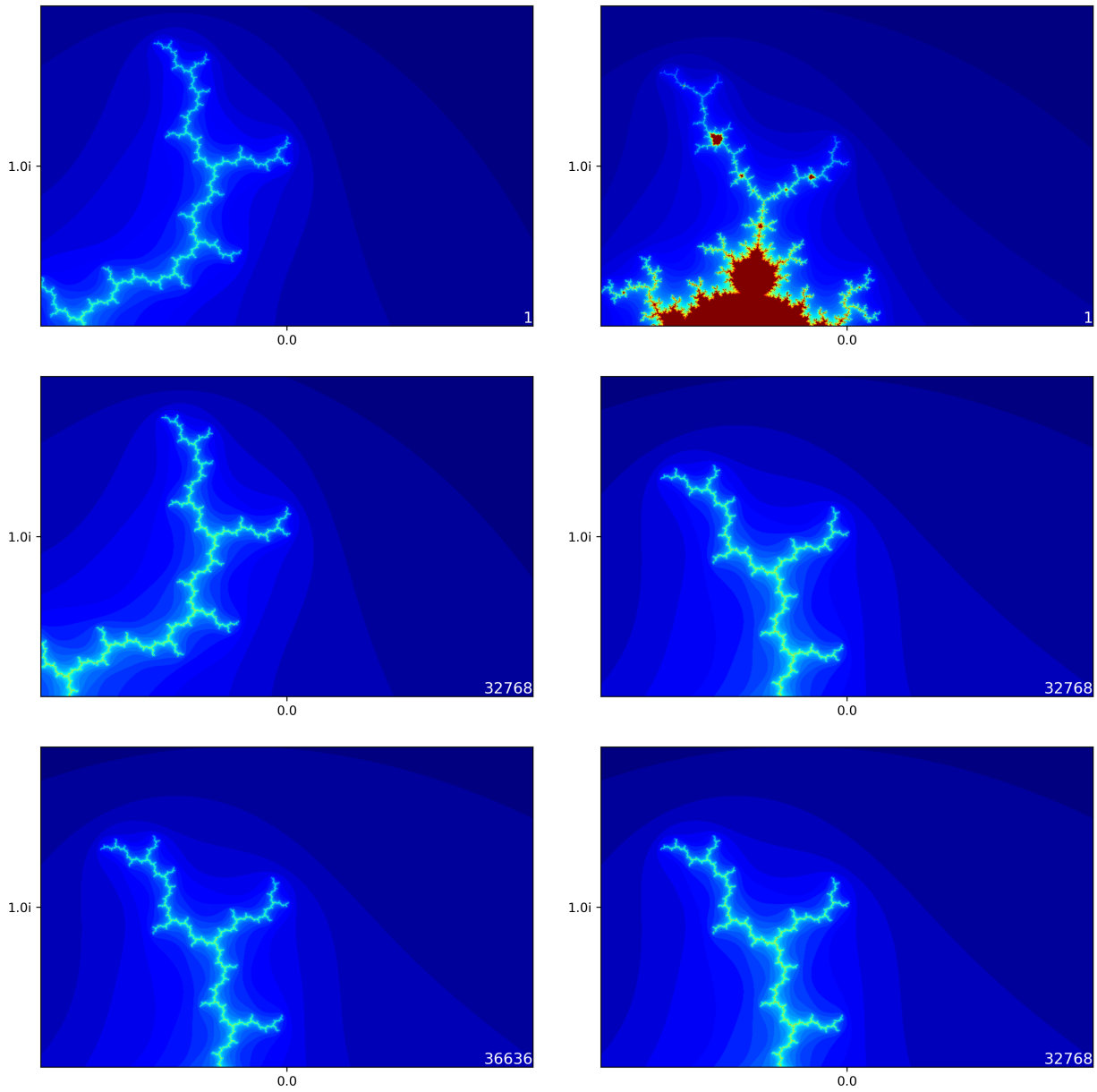

Figure 8: Zooming in at the Julia set J_i (left) and the Mandelbrot set (right) at the point i. The second row is zoomed in with a factor 5.66^7 compared to the first row, with a rotation of $7 \cdot 45°$. The images of the first row show the region $[-0.3, 0.3] \times [1 - 0.2, 1 + 0.2]i \subset \mathbb{C}$. The third row of images is a repetition of the second row, where the Julia set is multiplied with $\lambda = 1 + \frac{1}{2}i \approx 1.12e^{i\theta}, \theta \approx 26.6°$.

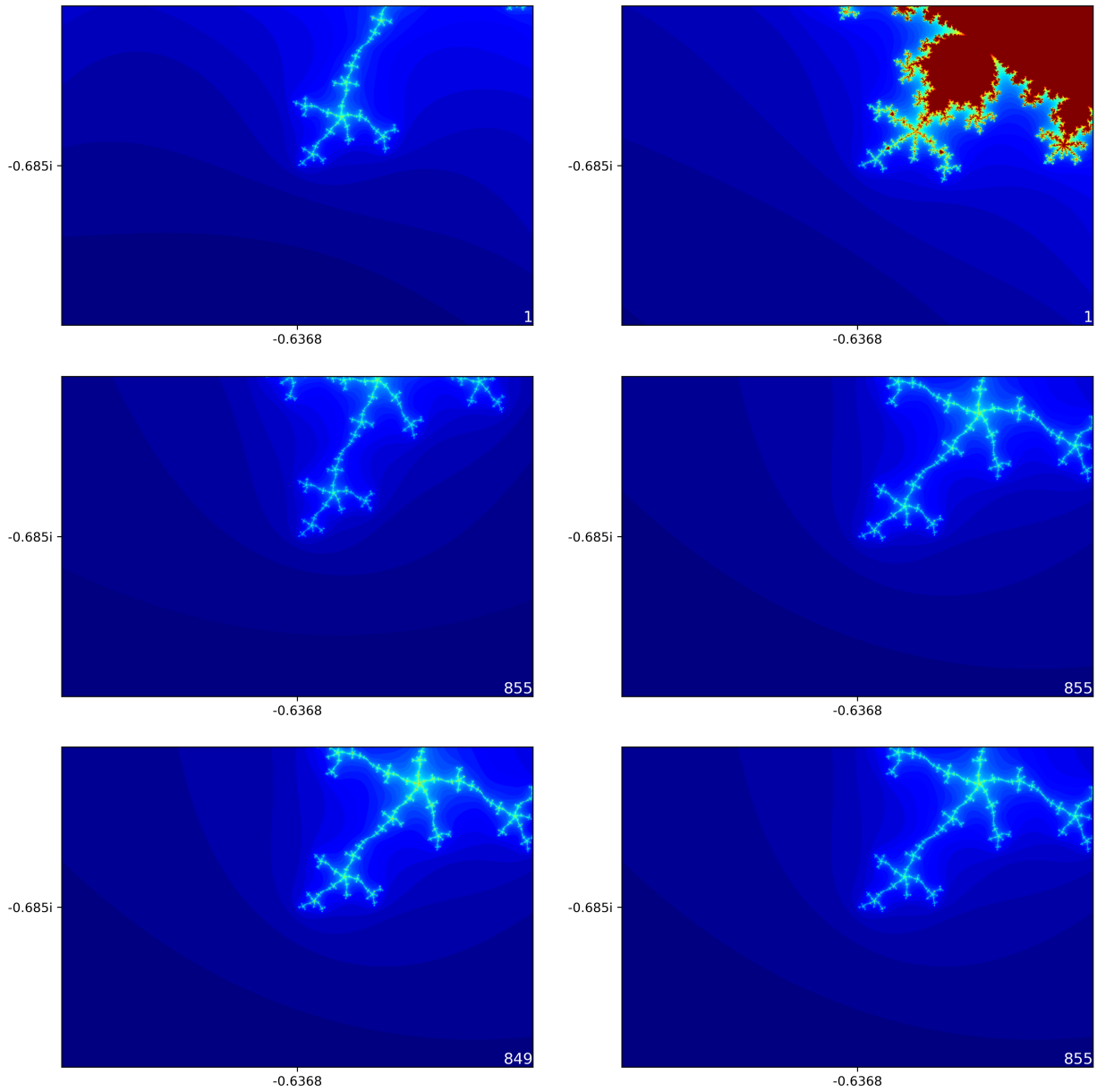

Figure 9: Zooming in at the Julia set J_c (left) and the Mandelbrot set (right) at the point $c \approx -0.6368 - 0.685i$. The second row is zoomed in with a factor 3.08^7 compared to the first row, with a rotation of $7 \cdot 12.82°$. The images of the first row show the region $[-0.6368 - 0.3, -0.6368 + 0.3] \times [-0.685 - 0.2, -0.685 + 0.2]i \subset \mathbb{C}$. The third row of images is a repetition of the second row, where the Julia set is multiplied with $\lambda \approx 0.9468 - 0.3006i \approx 0.9934e^{i\theta}, \theta \approx -17.6140°$.