Dynamics of continued fractions and kneading sequences of unimodal maps

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Credits

Joint work with C. Carminati (Pisa), C. Bonanno (Pisa), S. Isola (Camerino)

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Thanks to C. McMullen

Summary

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2. The exceptional set $\mathcal{E}$

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5. Univoque numbers

Gauss map

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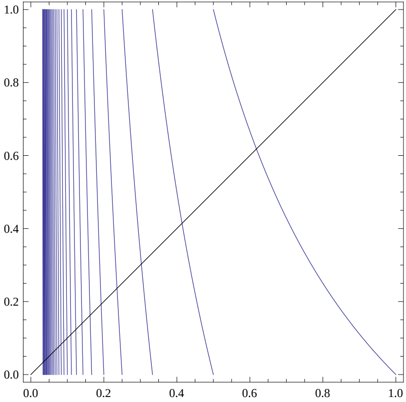

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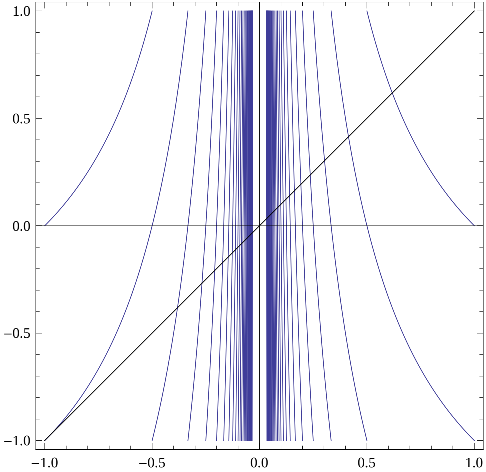

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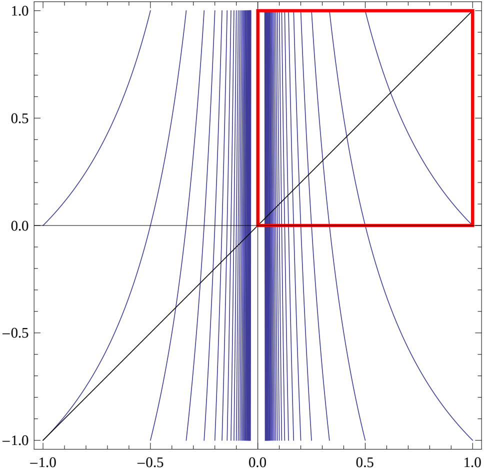

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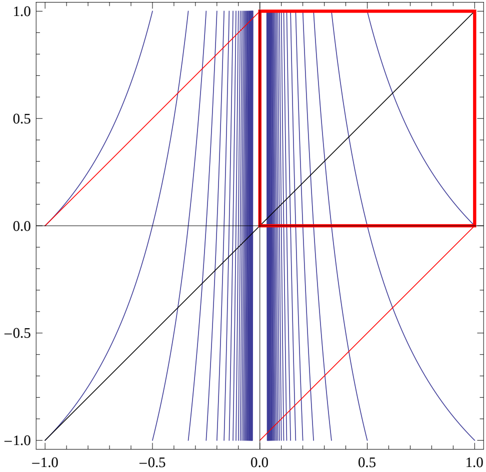

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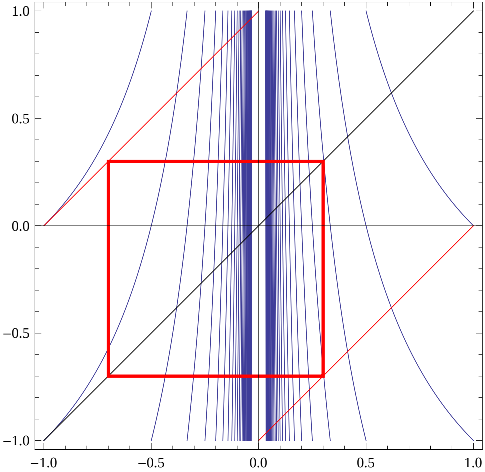

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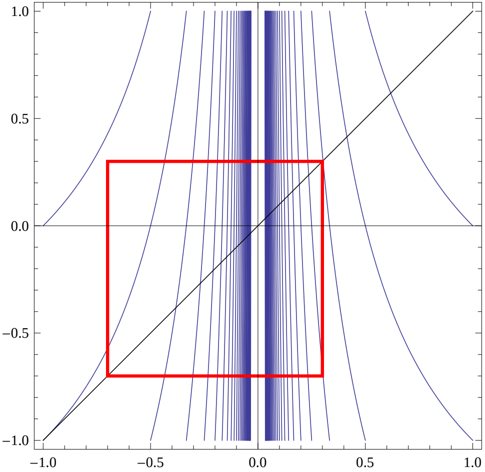

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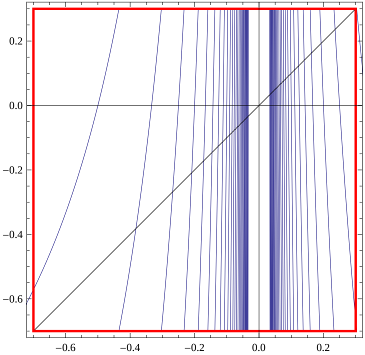
The maps $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ are defined as follows:

$$T_\alpha(x) := \frac{1}{|x|} - c_\alpha(x), \quad c_\alpha(x) := \lfloor \frac{1}{|x|} + 1 - \alpha \rfloor.$$



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and generate the α-continued fraction expansion:

$$x = \cfrac{\epsilon_{1,\alpha}}{c_{1,\alpha} + \cfrac{\epsilon_{2,\alpha}}{c_{2,\alpha} + \ddots}} \quad c_{n,\alpha} \in \mathbb{N}^+, \epsilon_{n,\alpha} \in \{\pm 1\}$$

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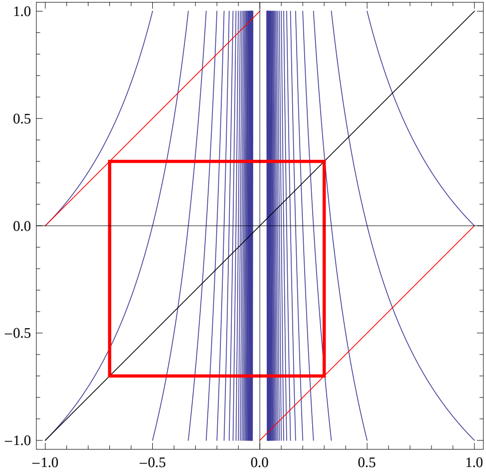

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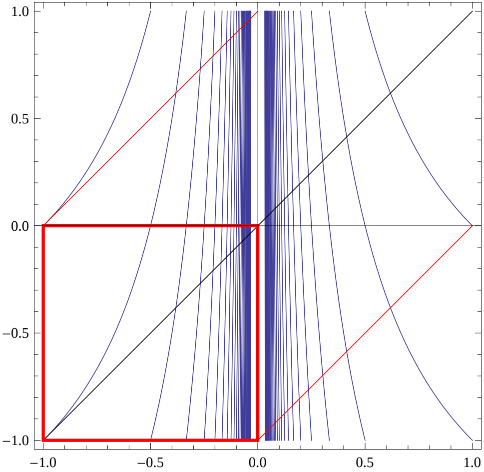

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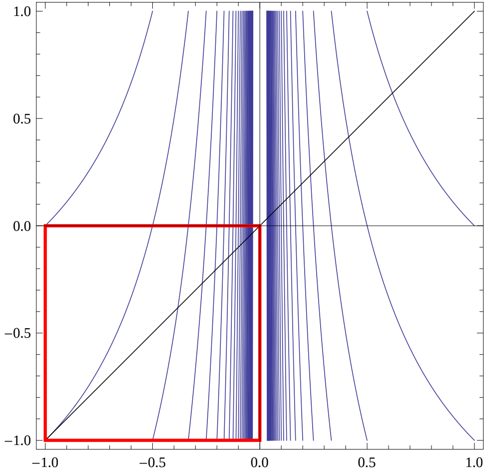

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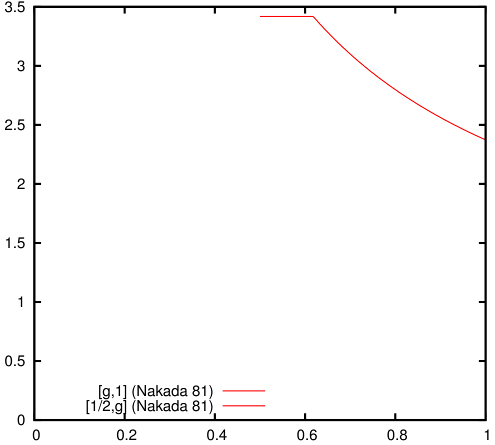
- T_α is ergodic w.r.t. Lebesgue measure;
- For almost every $x \in [0, 1]$, the entropy of T_α is

$$h(T_\alpha) = \lim_{n \to +\infty} \frac{2}{n} \log q_{n,\alpha}(x)$$

where $p_{n,\alpha}(x)/q_{n,\alpha}(x)$ is the n-th convergent of the α-expansion of x

The entropy function $\alpha \mapsto h(T_\alpha)$



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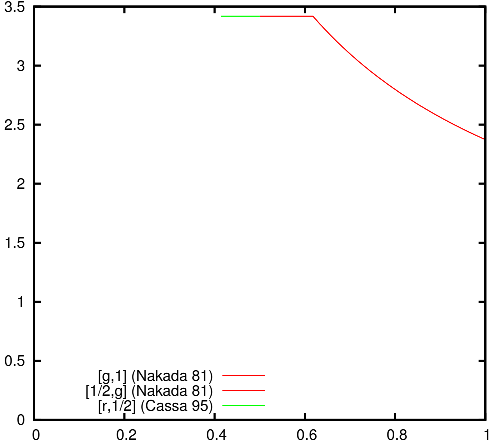

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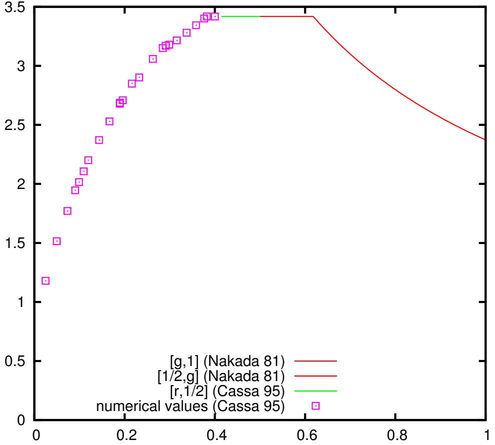

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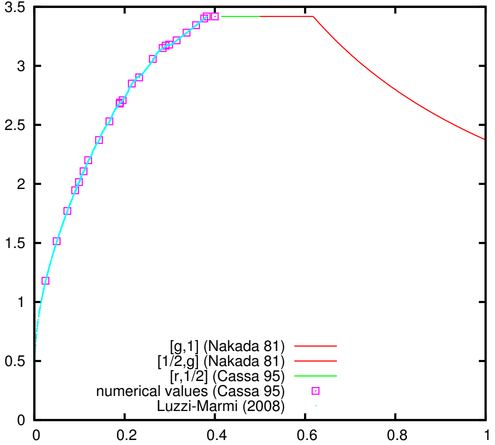
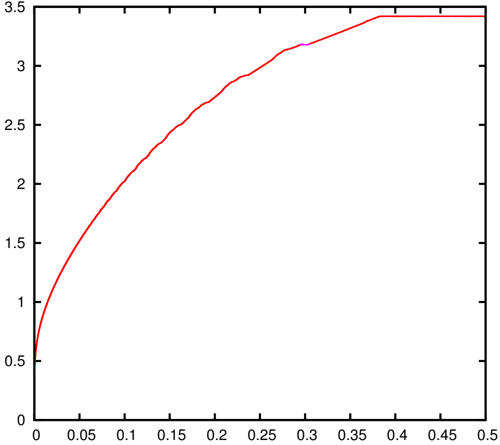


Zooming in



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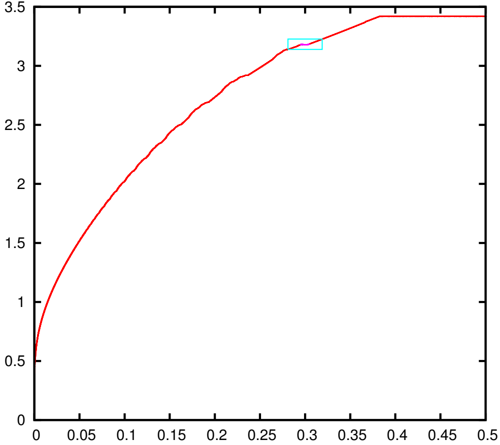

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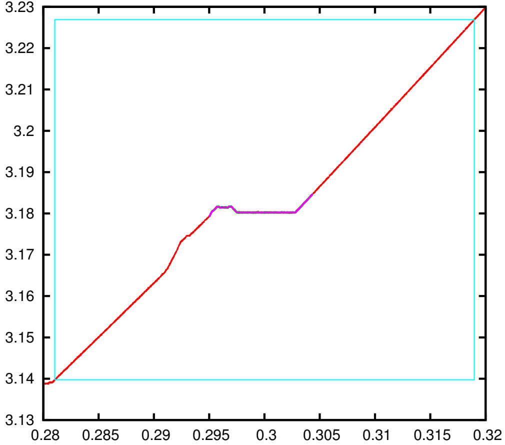

3.23
3.22
3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
3.13
0.28
0.285
0.29
0.295
0.3
0.305
0.31
0.315
0.32

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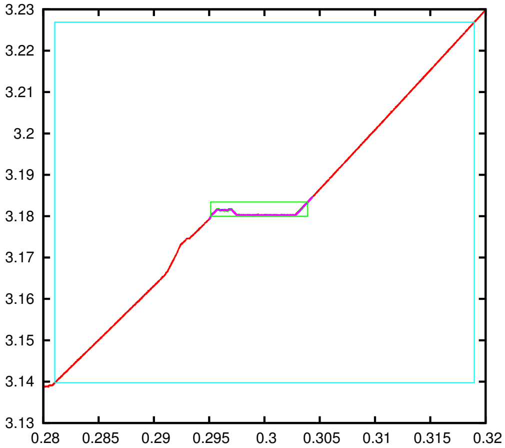
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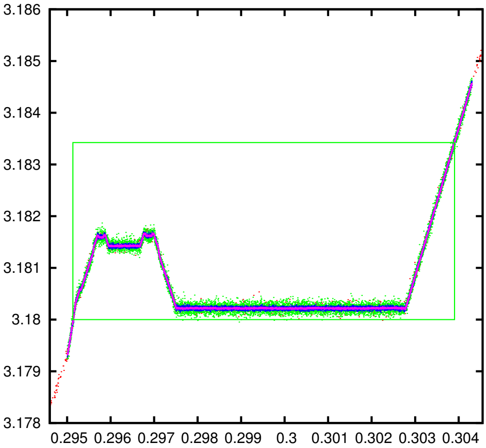

3.186
3.185
3.184
3.183
3.182
3.181
3.18
3.179
3.178
0.295 0.296 0.297 0.298 0.299 0.3 0.301 0.302 0.303 0.304

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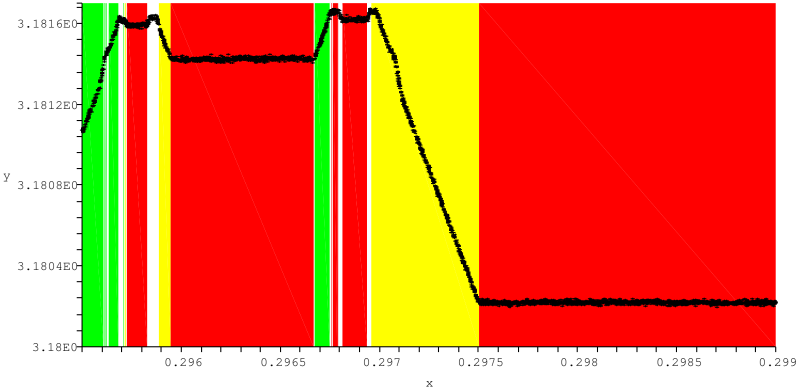

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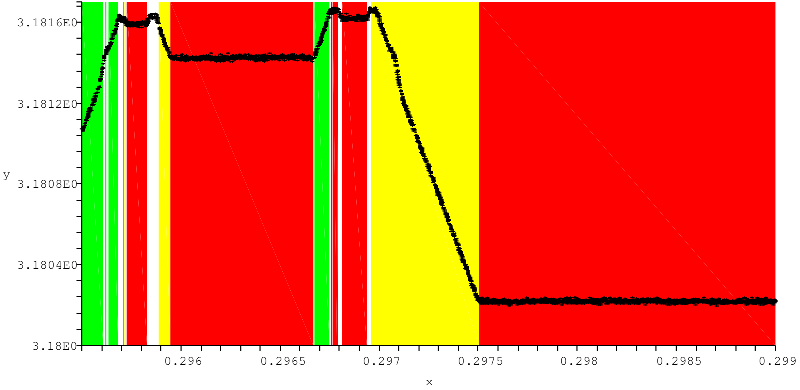

[NN] H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity **21** (2008), 1207-1225

A quick account of Nakada-Natsui's results

Nakada and Natsui defined *matching intervals* as intervals on which a condition of type

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Theorem (CT)

There exists a closed fractal set $\mathcal{E} \subset [0,1]$ *such that*

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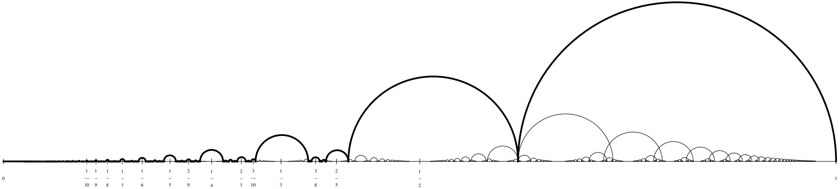
What is the underlying structure?

The exceptional set $\mathcal{E}$



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Using such strings we can construct the two quadratic irrationals
$\alpha^- := [0; \overline{A^-}]$

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The interval $I_a := (\alpha^-, \alpha^+)$ will be called the *quadratic interval* generated by $a \in \mathbb{Q} \cap (0, 1)$.

Thickening $\mathbb{Q}$

$$\mathcal{M} = \bigcup_{a \in \mathbb{Q} \cap]0,1]} I_a.$$

- $\mathcal{M}$ is an open neighbourhood of $\mathbb{Q} \cap]0,1]$;
- the connected components of $\mathcal{M}$ are quadratic intervals;

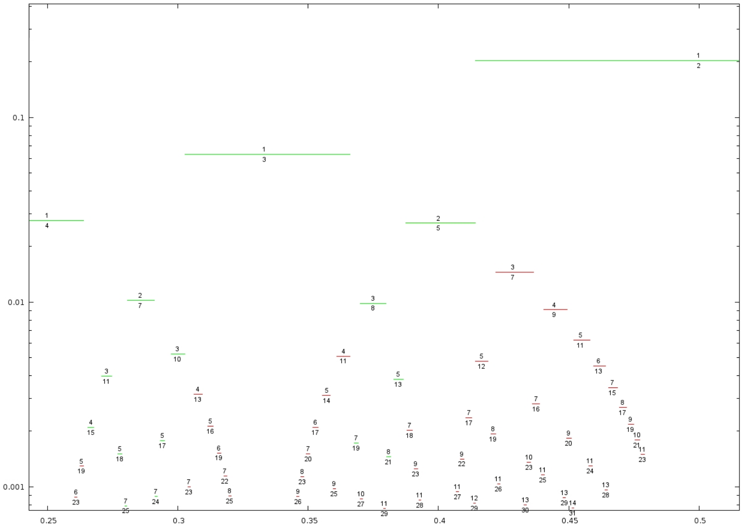
The *exceptional set*

$$\mathcal{E} := [0,1] \setminus \mathcal{M} = [0,1] \setminus \bigcup_{a \in \mathbb{Q} \cap]0,1]} I_a$$

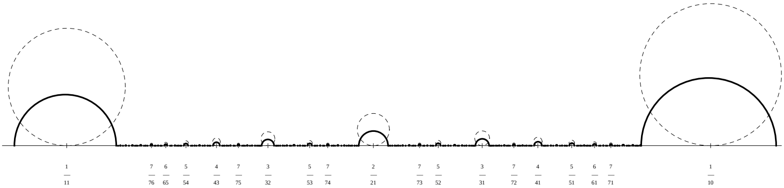
is such that

- $|\mathcal{E}| = 0$;
- $\dim_{\mathcal{H}}(\mathcal{E}) = 1$;
- $\mathcal{E} \setminus \{\text{isolated points}\}$ is a Cantor set;

Quadratic intervals



$\mathcal{E}$ vs horoballs



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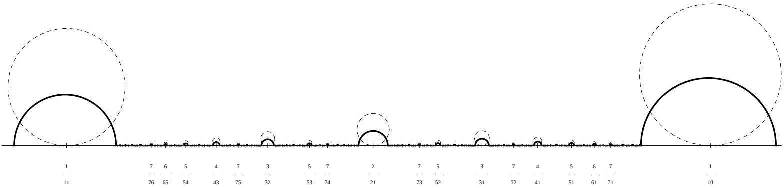

$\mathcal{E} \subseteq \{$numbers of bounded type$\} \Rightarrow$ measure zero

Direct description of $\mathcal{E}$.

Proposition (Bonanno, Carminati, Isola, T, 2010)

$$\mathcal{E} = \{x \in [0,1] \ : \ G^k(x) \geq x, \ \forall k \in \mathbb{N}\}.$$

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(G denotes the Gauss map)

Symbolic dynamics of unimodal maps

Let $f : [0, 1] \to [0, 1]$ be a smooth map, F is called *unimodal* if it has exactly one critical point $0 < c_0 < 1$ and $f(0) = f(1) = 0$.

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$$i(x) = s_1 s_2 \dots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{if } f^{i-1}(x) < c_0 \\ 1 & \text{if } f^{i-1}(x) \geq c_0 \end{cases}$$

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Theorem (Milnor-Thurston '77)

The kneading sequence determines the topological entropy. The entropy of the quadratic family $f(x) = \lambda x(1 - x)$ is monotone in λ.

The set of all kneading sequences Λ

Using the kneading sequence one can produce a *kneading invariant* τ_f

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Proposition

The set of all kneading invariants of all real quadratic polynomials is

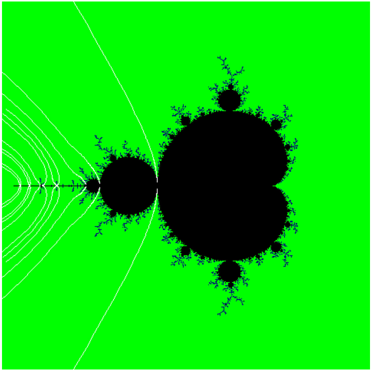
$$\Lambda := \{x \in [0,1] : T^k(x) \leq x \ \forall k \in \mathbb{N}\}$$

where T is the classical tent map.

Λ vs Mandelbrot

$$\Lambda := \{x \in [0,1] : T^k(x) \leq x \;\forall k \in \mathbb{N}\}$$

Λ corresponds to the set of external rays which 'land' on the bifurcation locus of the real quadratic family, i.e. the real slice of the Mandelbrot set.



Identity of bifurcation sets

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$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \mapsto \varphi(x) = 0.\underbrace{11\ldots1}_{a_1}\underbrace{00\ldots0}_{a_2}\underbrace{11\ldots1}_{a_3}\ldots$$

is an orientation-reversing homeomorphism which maps $\mathcal{E}$ onto $\Lambda \setminus \{0\}$.

Minkowski's question mark function

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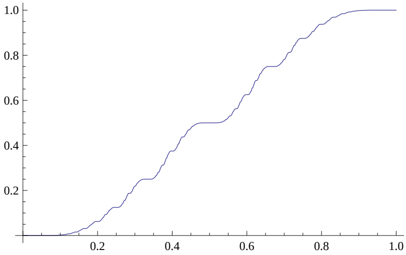
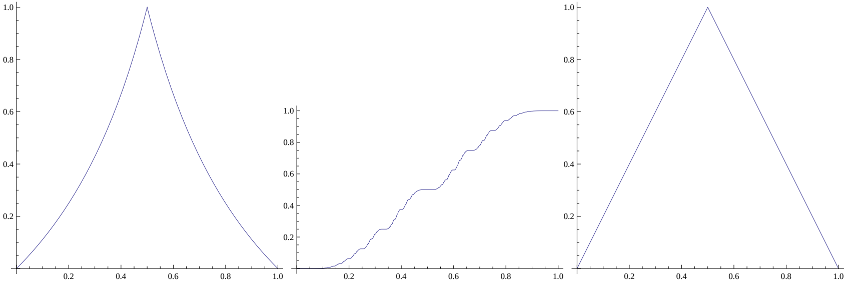


The dictionary

Continued fractions	$\Leftrightarrow$	**Binary expansions**
$\mathcal{E}$	$\leftarrow ? \rightarrow$	Λ

From Farey to the tent map, via ?

Minkowski's question-mark function conjugates the Farey map with the tent map



A unified approach

The dictionary yields a unified proof of the following results:

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$$H.dim(\partial\mathcal{M} \cap \mathbb{R}) = 1$$

(Zakeri, 2000)

Univoque numbers

Let us fix $1 < \beta < 2$ and consider the β-expansion of 1

$$1 = \frac{\epsilon_1}{\beta^1} + \frac{\epsilon_2}{\beta^2} + \dots$$

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Theorem (Allouche-Cosnard + ϵ)

Binary expansions of univoque numbers are in bijection with nonperiodic elements of Λ:

$$\mathcal{U} = \Lambda \setminus \mathbb{Q}_1$$

where $\mathbb{Q}_1$ is the set of rational numbers with odd denominator.

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The dictionary

Continued fractions	$\Leftrightarrow$	**Binary expansions**
$\mathcal{E}$	$\leftarrow ? \rightarrow$	Λ
$\alpha -$ continued fractions		unimodal maps
numbers of generalized bounded type (CT, 2011)		external rays on Julia sets
cutting sequences for geodesics on torus (Cassaigne, 1999)		univoque numbers

The end

Thank you!

References

C.Carminati, S.Marmi, A.Profeti, G.Tiozzo: *The entropy of alpha-continued fractions: numerical results*
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