Hitting measures for random walks on Fuchsian groups - The fundamental inequality

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University of Toronto

Enriques-Lebesgue seminar
January 18, 2021

Summary

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2. History and questions

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joint with Petr Kosenko

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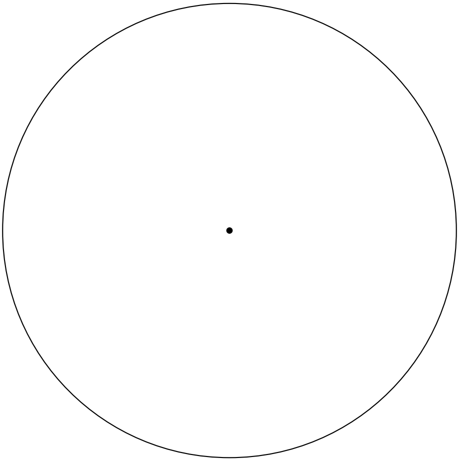

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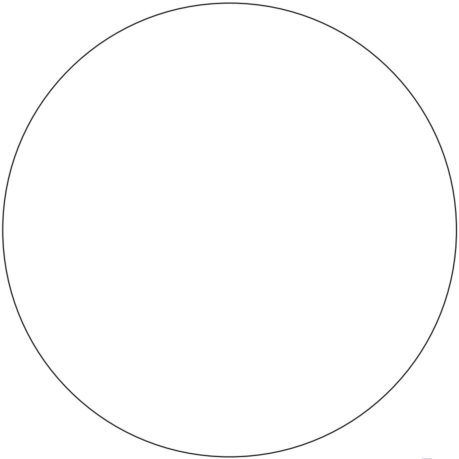

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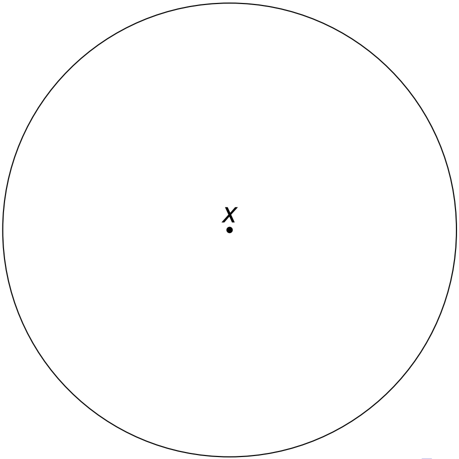

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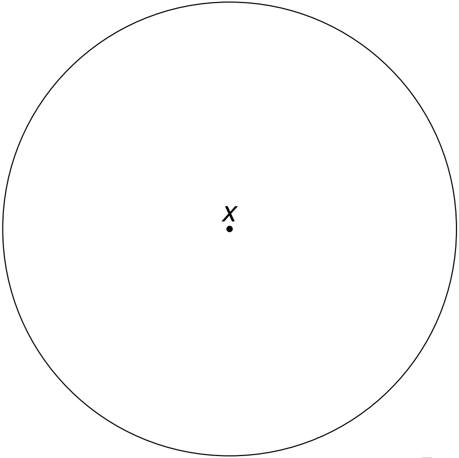

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Consider the **sample path**

$$w_n := g_1 g_2 \dots g_n$$

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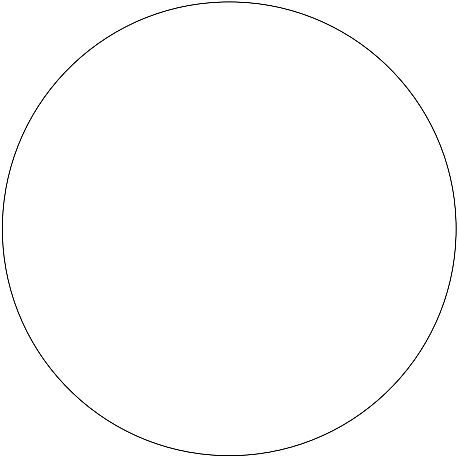

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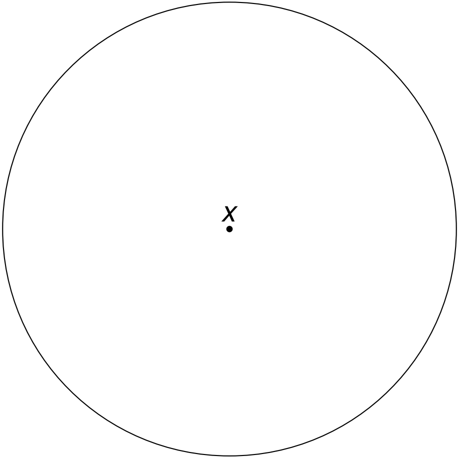

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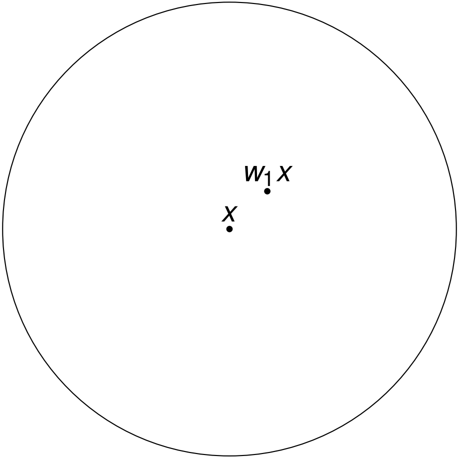

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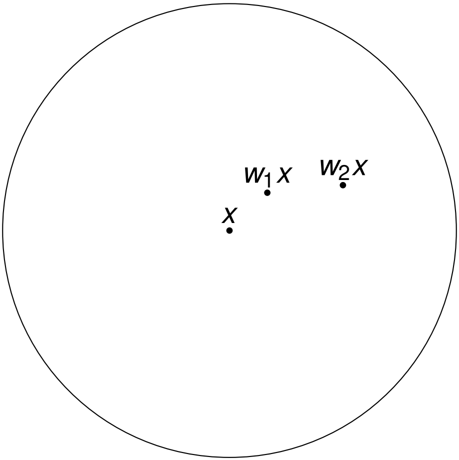

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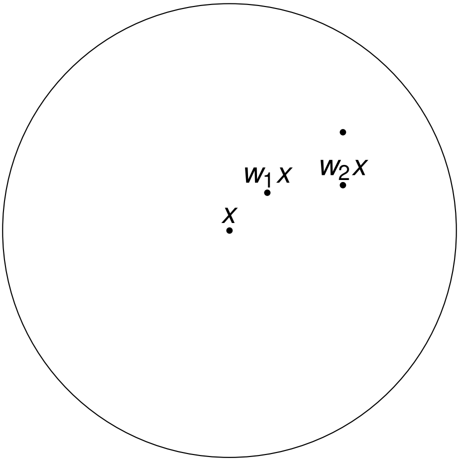

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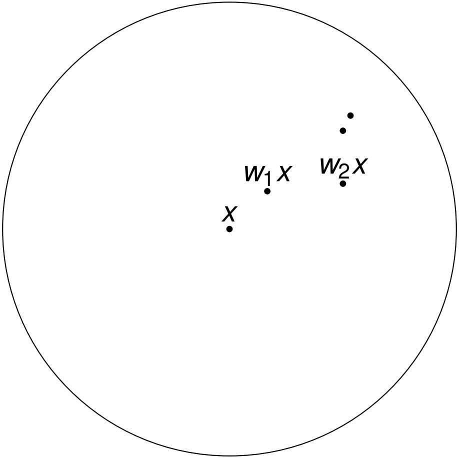

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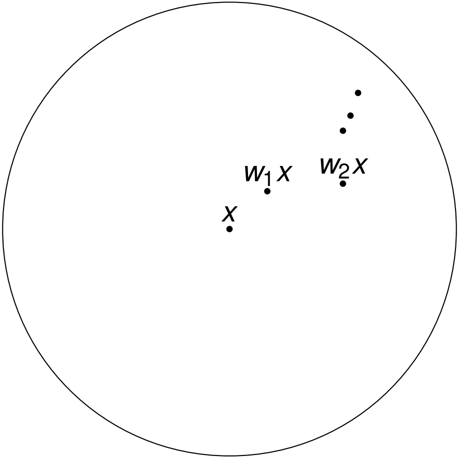

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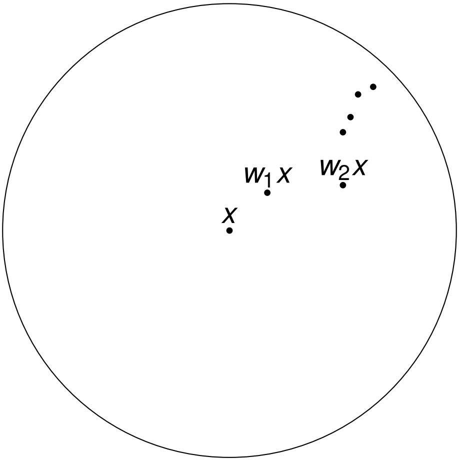

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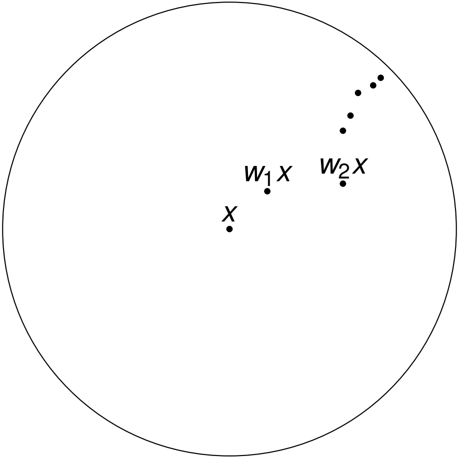

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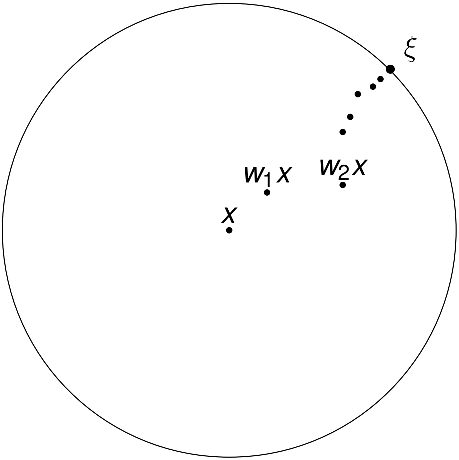
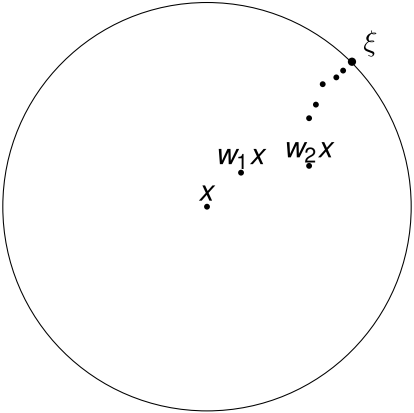


Boundary measures



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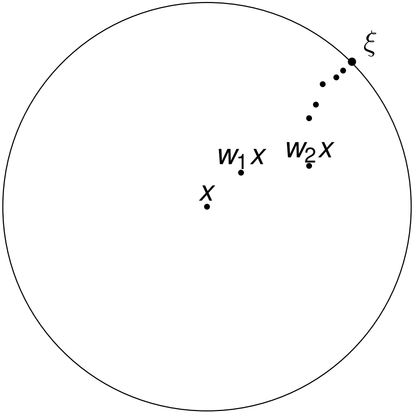

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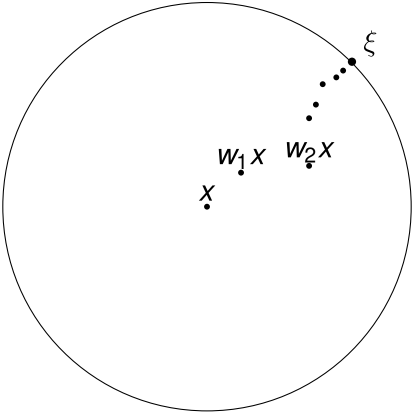

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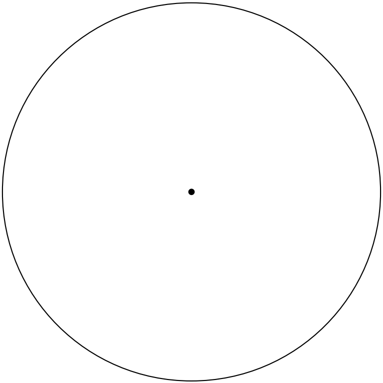

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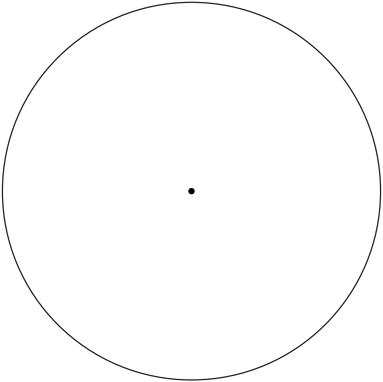

Theorem (Poisson representation)

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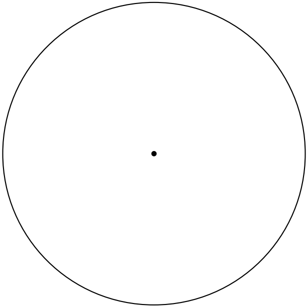

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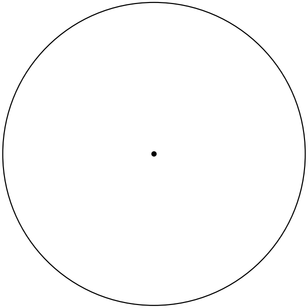

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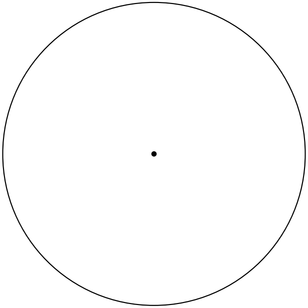

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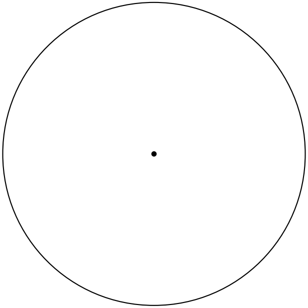

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Question. Can these two measures be the same, or in the same measure class?

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  Guivarc'h-LeJan '90, Deroin-Kleptsyn-Navas '09, Blachére-Haissinsky-Mathieu '11, Randecker-T. '19

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Conjecture (Kaimanovich-Le Prince '11)

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Every <u>discrete</u>, finitely supported measure on $SL_2(\mathbb{R})$ *is singular at infinity.*

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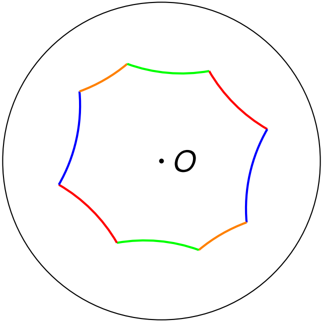

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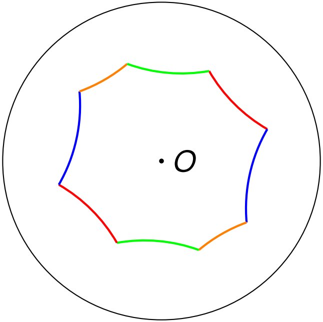

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Let P be a symmetric hyperbolic polygon with $2m$ sides, with sum of angles 2π. Let μ be a symmetric prob. measure supported on the set $(t_i)_{i=1}^{2m}$ of translations which identify opposite sides. Then the hitting measure ν_μ is singular w.r.t. Lebesgue measure.

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- *$h < \ell$ is strict;*
- *The Hausdorff dimension satisfies*

$$\text{H.dim } \nu_\mu = \frac{h}{\ell} < 1.$$

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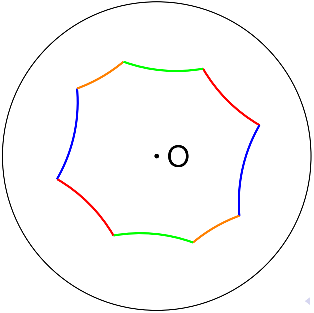

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4. *For any $o \in \mathbb{H}^2$, there exists a constant $C > 0$ such that for any $g \in \Gamma$ we have*

$$|d_\mu(1, g) - d_{\mathbb{H}}(o, go)| \leq C.$$

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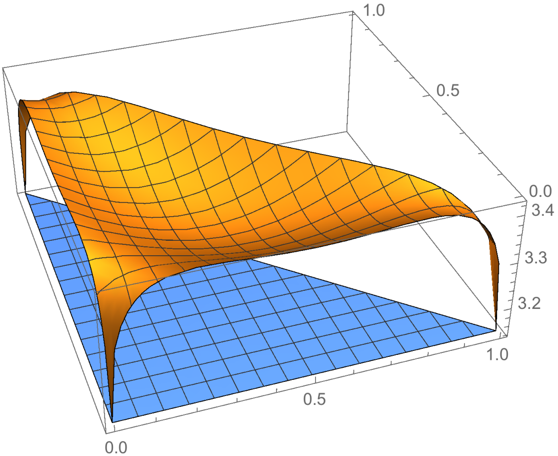

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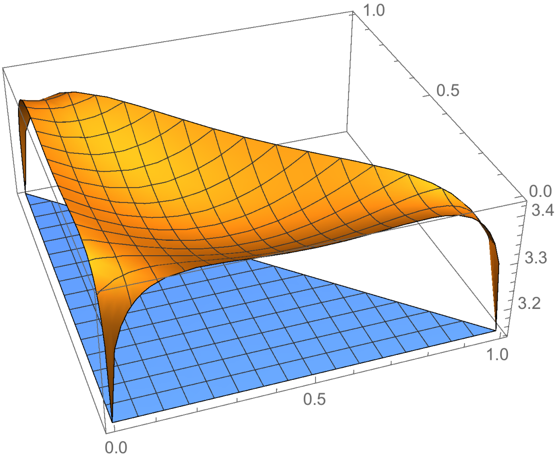

The graph of $f(x) := \sum_{i=1}^{3} \arccos((1 - x_i)(1 - x_{i+1}))$ subject to the constraint $\sum_{i=1}^{3} x_i = 1$, compared with the constant function at height π.

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The end

Thank you!
Grazie!
Merci!