Tuning and plateaux for the entropy of α-continued fraction transformations

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Credits

Joint work with C. Carminati (Pisa)

Summary

1. α-continued fractions

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2. The entropy function $h(\alpha)$

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6. Local monotonicity of the entropy

Euclid's algorithm and continued-fractions

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q &= a_1 r_0 + r_1
\end{aligned}
$$

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INFINITE EXPANSION

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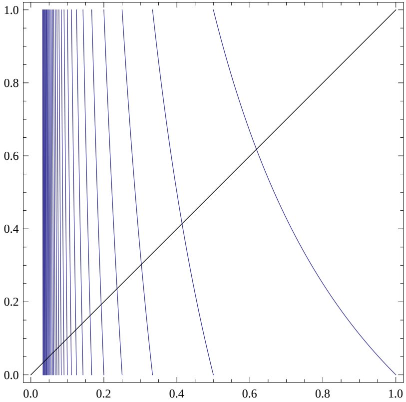

Nakada's α-continued fraction transformations

For each $\alpha \in [0, 1]$, we can define a α-euclidean algorithm, where we take the remainder to be in $[\alpha - 1, \alpha]$.
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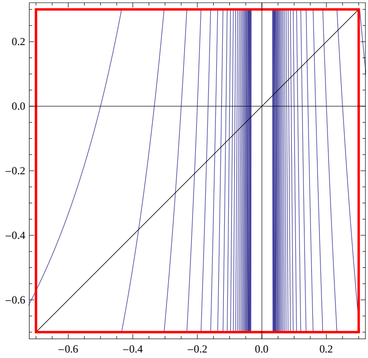

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and associated to the α-continued fraction expansion:

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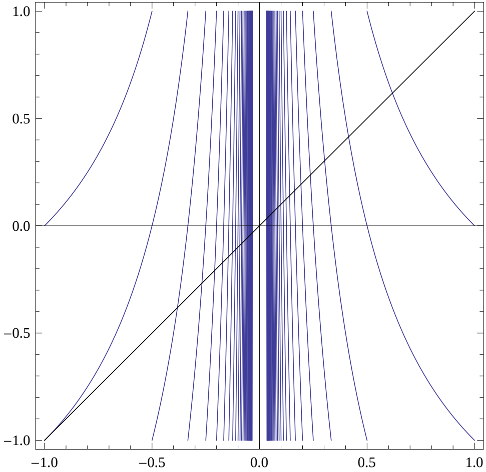

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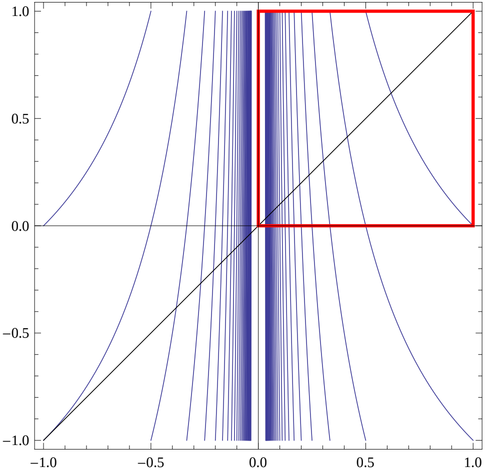

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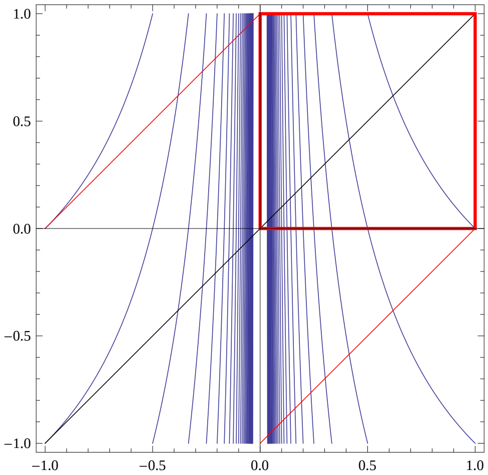

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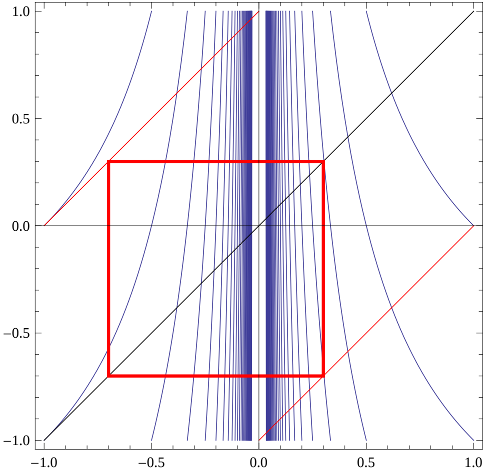

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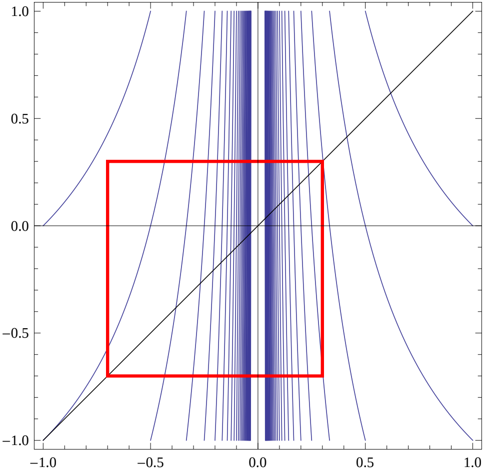

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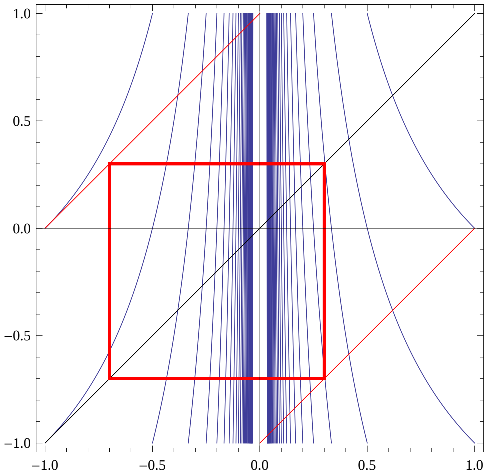

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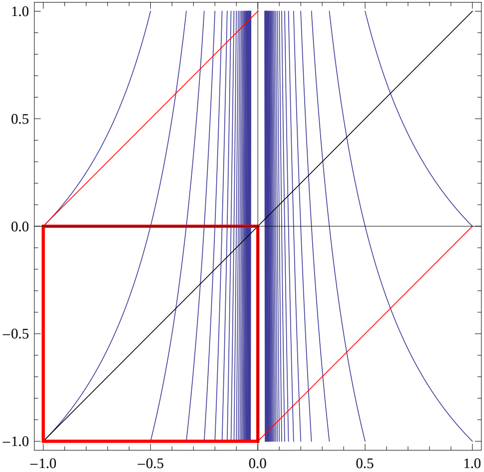

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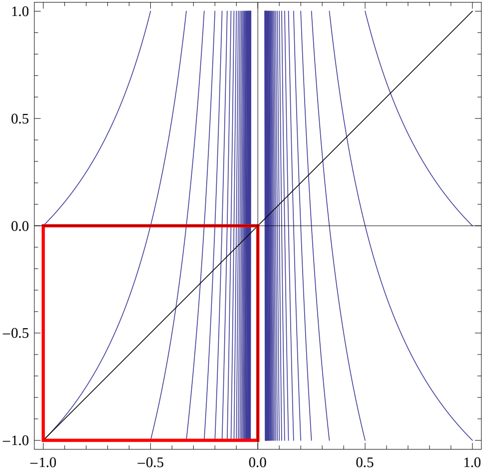

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- the speed of convergence of the α-euclidean algorithm: The average number of steps over all rationals of denominator less than N is

$$P_N(\alpha) \cong \frac{2}{h(\alpha)} \log N$$

[Bourdon-Daireaux-Vallée]

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- the growth rate of the denominators : For almost every $x \in [0, 1]$

$$h(\alpha) = \lim_{n \to +\infty} \frac{2}{n} \log q_{n,\alpha}(x)$$

  where $p_{n,\alpha}(x)/q_{n,\alpha}(x)$ is the n-th convergent of the α-expansion of x

Entropy

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It measures:

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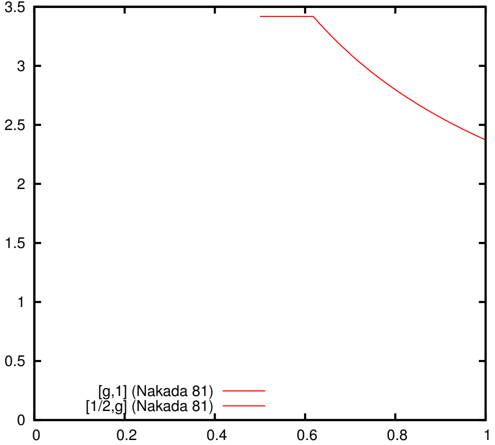
- the growth rate of the denominators
- how chaotic the map T_α is

The entropy function $\alpha \mapsto h(T_\alpha)$



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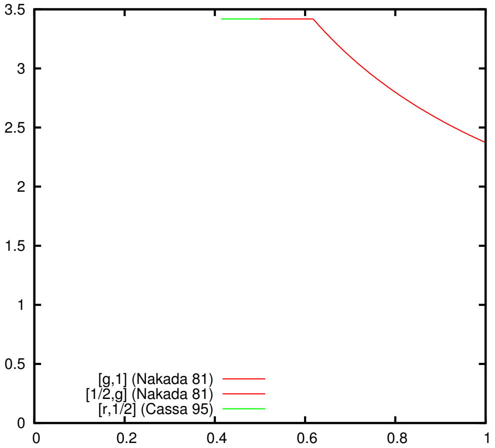

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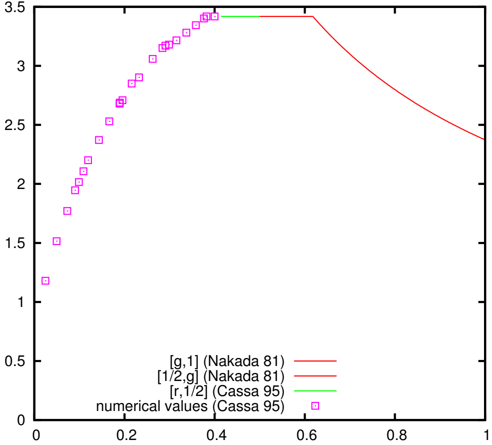

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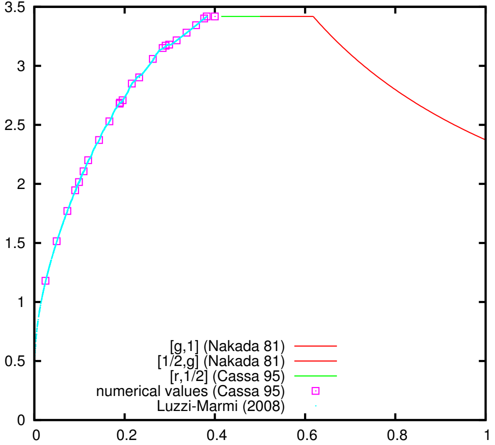
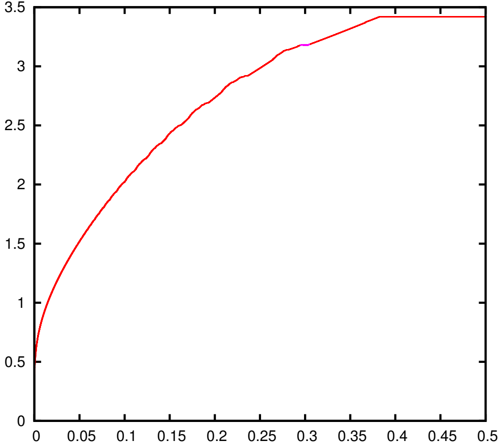
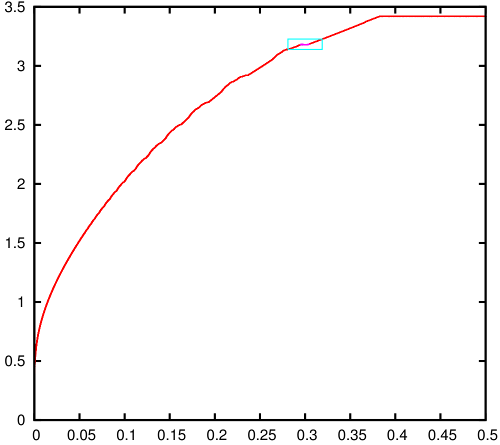


Zooming in

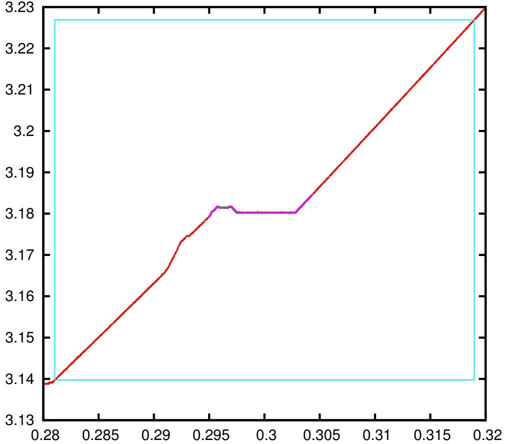


Is entropy monotone increasing for $\alpha < \frac{1}{2}$?

Zooming in

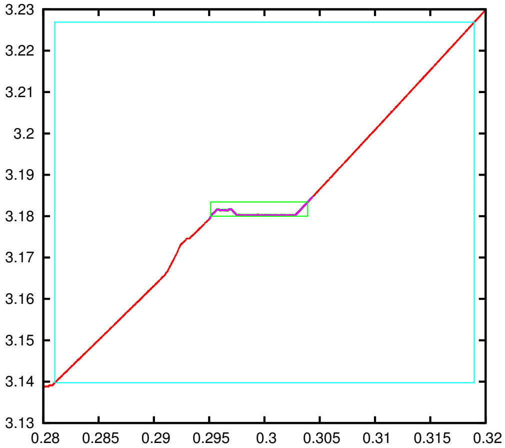


Zooming in

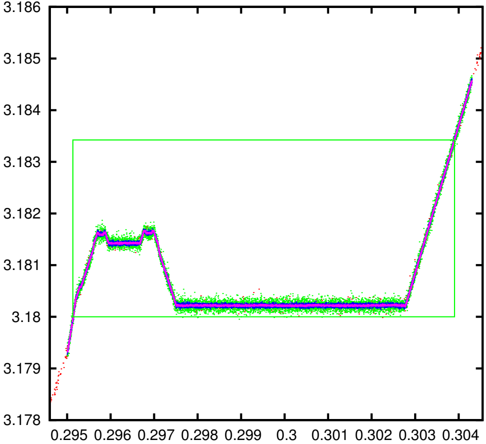


No, it is not monotone!

Zooming in


3.23
3.22
3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
3.13
0.28
0.285
0.29
0.295
0.3
0.305
0.31
0.315
0.32

Zooming in



It seems like entropy displays a fractal structure

Global behaviour of $h(\alpha)$

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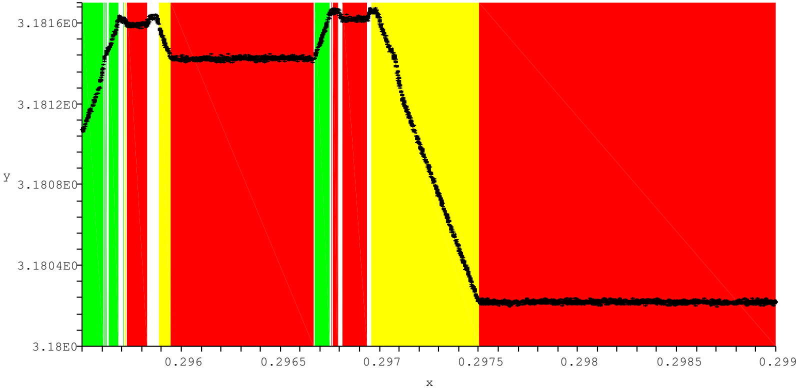
How to describe and explain the fractal structure?



y
3.1816E0
3.1812E0
3.1808E0
3.1804E0
3.18E0
0.296
0.2965
0.297
0.2975
0.298
0.2985
0.299
x

Matching, a dynamical source of monotonicity

Nakada and Natsui defined *matching intervals* as intervals on which the orbits of the two endpoints collide:

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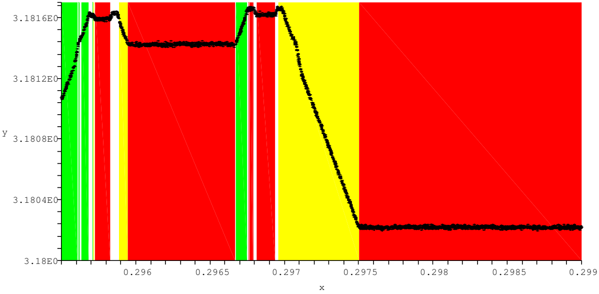

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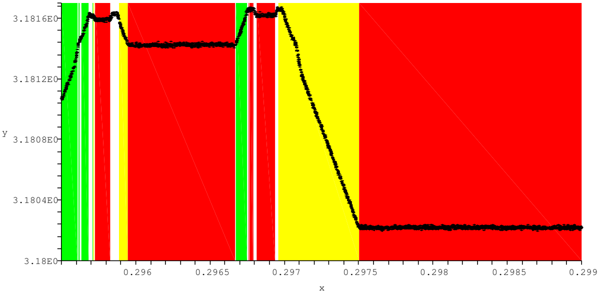

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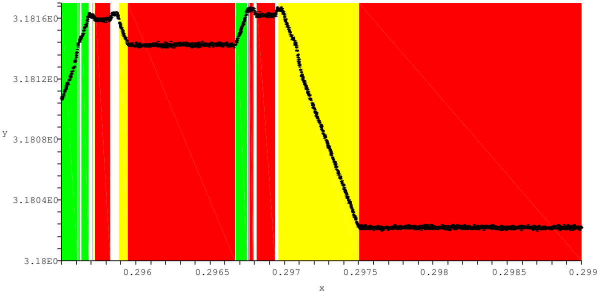

Conjecture

The union of all matching intervals is dense and has full measure in parameter space.

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Using such strings we can construct the two quadratic irrationals
$\alpha^- := [0; \overline{A^-}]$

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$$\frac{3}{10} = [0; 3, 3] = [0; 3, 2, 1].$$

So any $a \in \mathbb{Q} \cap (0, 1)$ will have two C.F. expansions of the type

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Quadratic intervals are matching intervals

Theorem (Carminati-T., 2010)

Let I_r be a maximal quadratic interval, and $r = [0; a_1, \ldots, a_n]$ with n even. Let

$$N = \sum_{i \text{ even}} a_i \qquad M = \sum_{i \text{ odd}} a_i \tag{1}$$

Then for all $\alpha \in I_r$,

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Corollary

The union of all matching intervals is dense of full measure.

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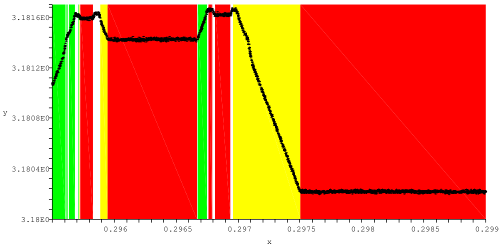

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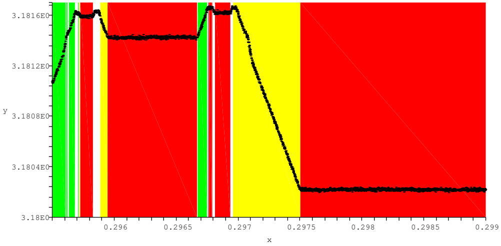

How about the fractal structure?

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Idea: τ_r maps the large scale structure to a smaller scale structure, thus creating the fractal self-similarity.

Results: self-similarity of parameter space

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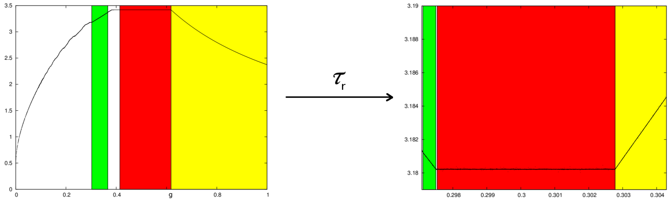

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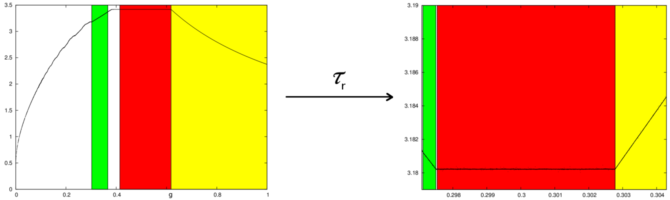

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Every plateau of h *is the interior of a neutral tuning window* W_r*.*

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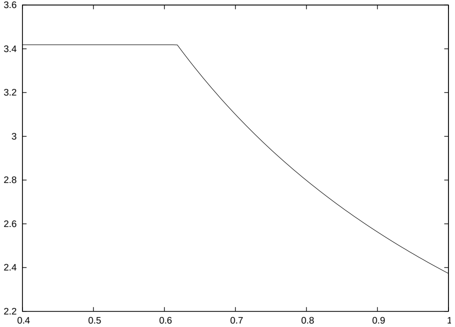

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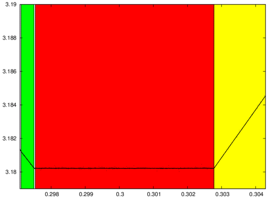

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   - (iii) otherwise, h has mixed monotonic behaviour at α, i.e. in every neighbourhood of α there are infinitely many intervals on which h is increasing, infinitely many on which it is decreasing and infinitely many on which it is constant.

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- 2.(iii) for a set of parameters of Hausdorff dimension 1!
- there is an explicit algorithm to decide which case occurs, given the usual continued fraction expansion of α.

The end

Thank you!

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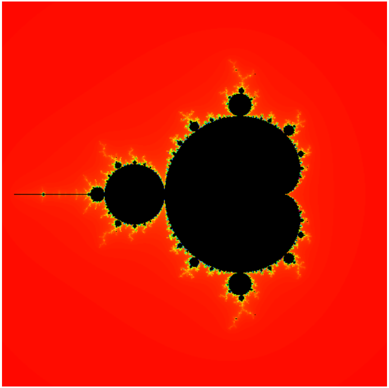
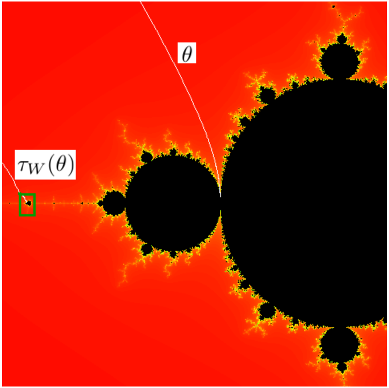


Substitutions and tuning

The Mandelbrot set has a self-similar structure. More precisely, there are baby copies of $\mathcal{M}$ everywhere near its boundary.



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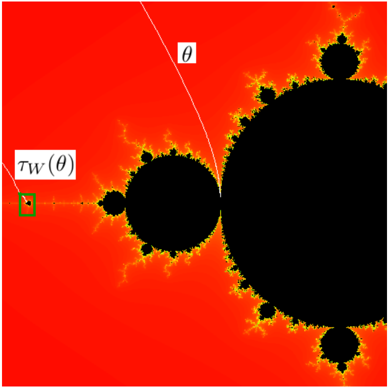

Baby copies are images of $\mathcal{M}$ via the Douady-Hubbard tuning maps τ_W.

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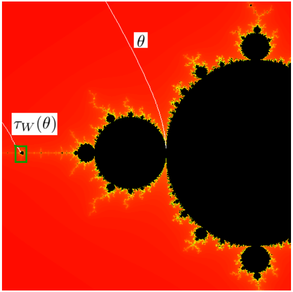

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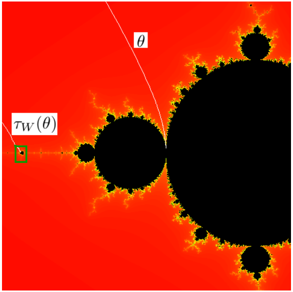

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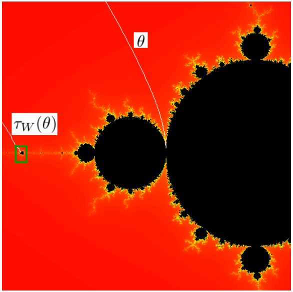

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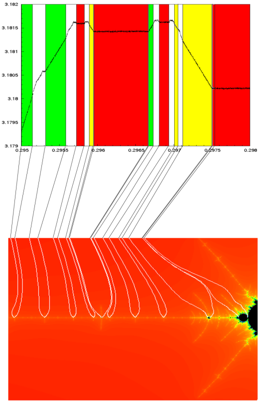
E.g.: Feigenbaum parameter $\Leftrightarrow$ Thue-Morse sequence!

Dictionary



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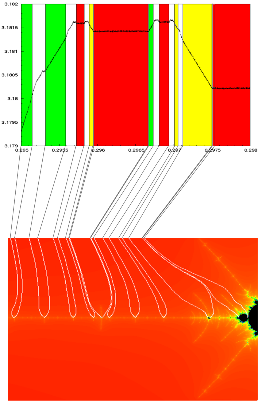

The set of rays landing on the real slice of the Mandelbrot set is isomorphic to the bifurcation set $\mathcal{E}$ for α-c.f. [Bonanno, Carminati, Isola, T., 2011] Hence the Douady-Hubbard substitution rule translates into our definition of tuning maps for α-c.f.!

The end

Thank you!