Random walks on weakly hyperbolic groups

Giulio Tiozzo
University of Toronto

Random and Arithmetic Structures in Topology
MSRI - Fall 2020

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Random walks, WPD actions, and the Cremona group

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Answer. It depends on the topography (geometry) of the city.

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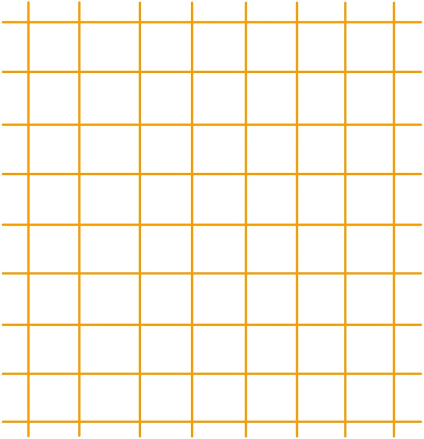

What is the probability of coming back to where you started?

Recurrence

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Exercise. Prove the Lemma.

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$\therefore$ our RW is recurrent.

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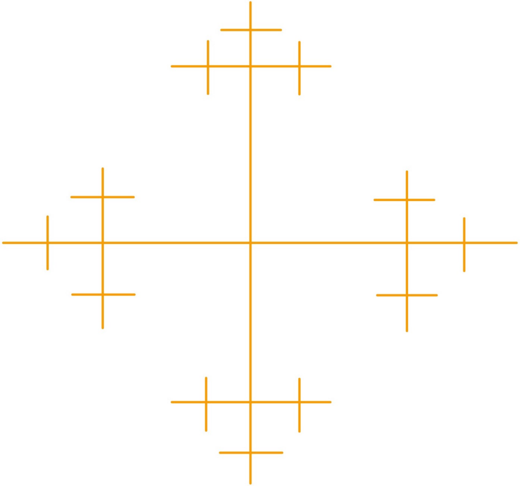
Exercise. Prove Polya's theorem for $d = 3$. Moreover, for the simple random walk on $\mathbb{Z}^d$, show that $p^{2n}(0,0) \approx n^{-\frac{d}{2}}$.

Transient random walks

Example 2: Tree City

In Tree City, the map has the shape of a 4-valent tree.



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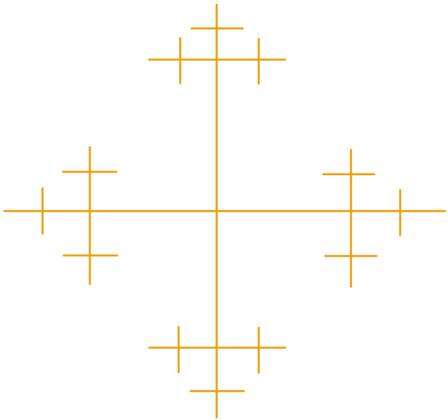

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(do we know $\lim_{n\to\infty} \frac{d_n}{n}$ exist?)

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If you fix a basepoint $x \in X$ you can look at the sequence $(w_n \cdot x) \subseteq X$.

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Moreover, we define the word metric or word distance between $g, h \in G$ as

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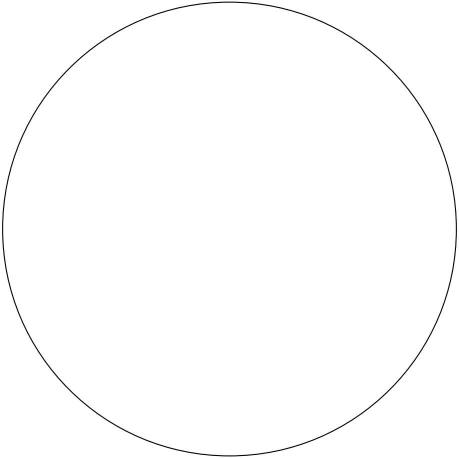

The disc has a natural topological boundary, i.e. the circle.
This RW converges a.s. to the boundary (Furstenberg).

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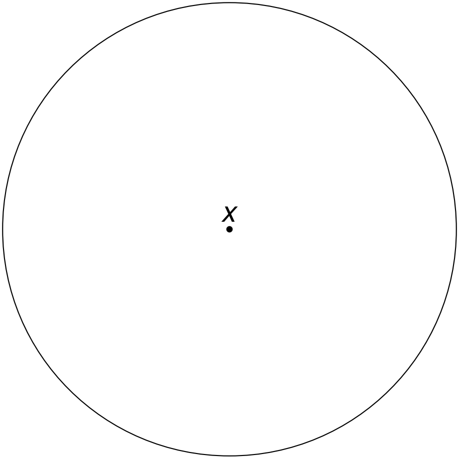

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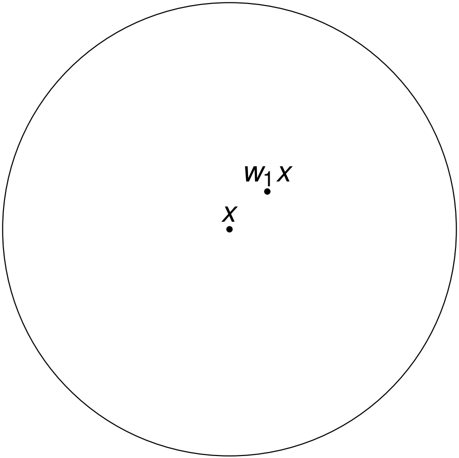

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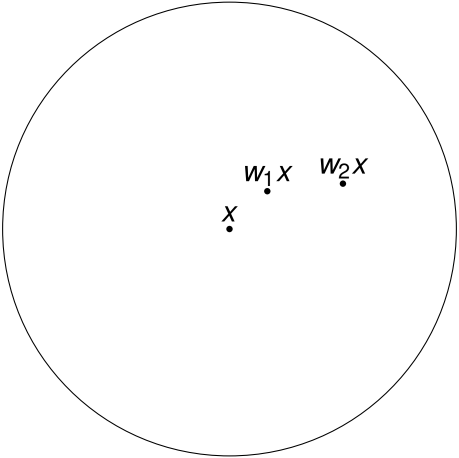

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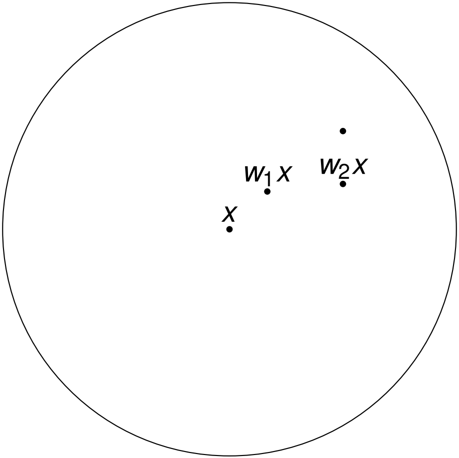

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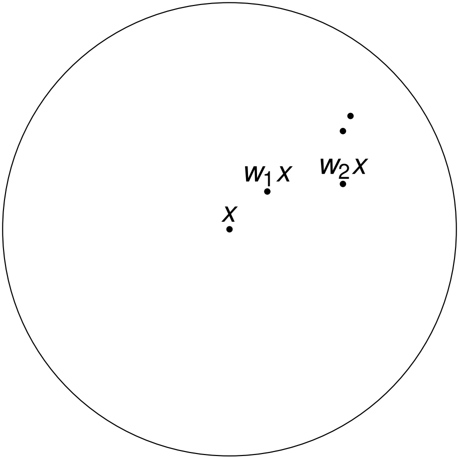

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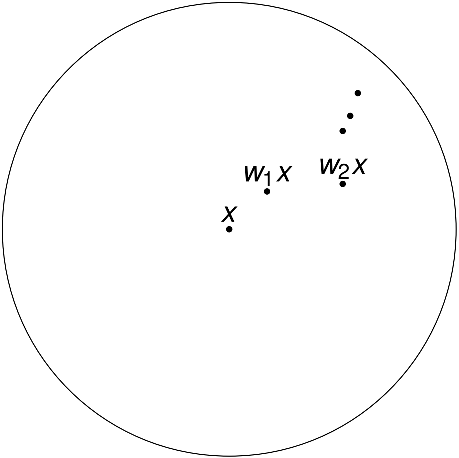

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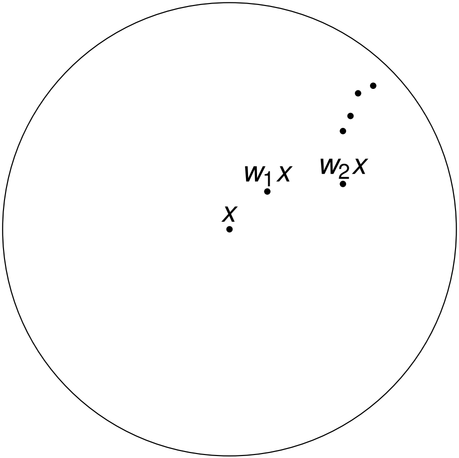

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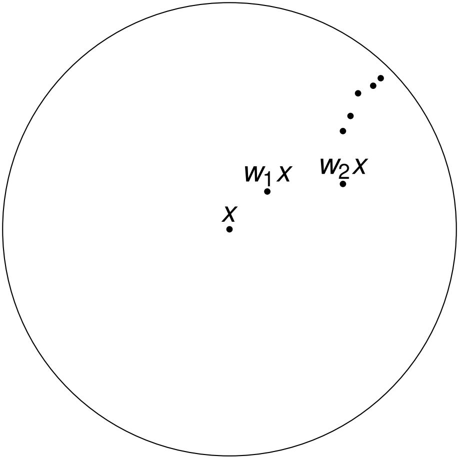

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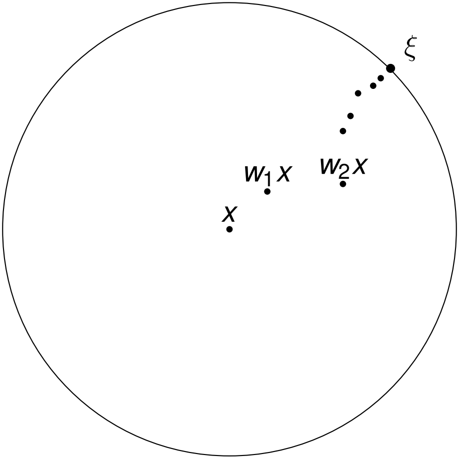

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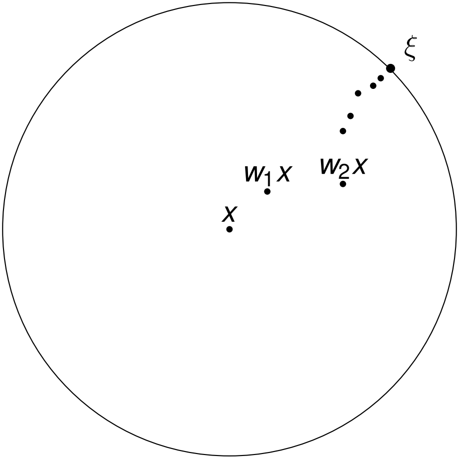

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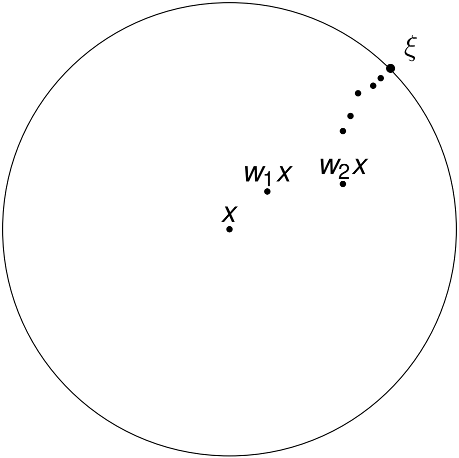

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A measure μ on G has finite first moment on X if for some (equivalently, any) $x \in X$

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Proof.

For any $x \in X$, the function $a(n,\omega) := d(x, w_n(\omega)x)$ is a subadditive cocycle, because

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where T is the shift on the space of increments, hence the claim follows by Kingman's subadditive ergodic theorem. $\square$

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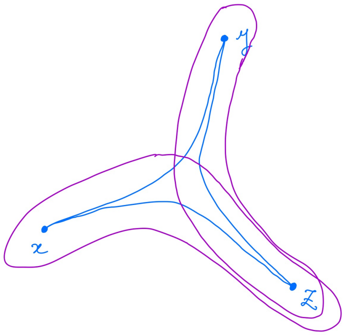

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Recall a space is proper if metric balls $\{z \in X \ : \ d(x, z) \leq R\}$ are compact.

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