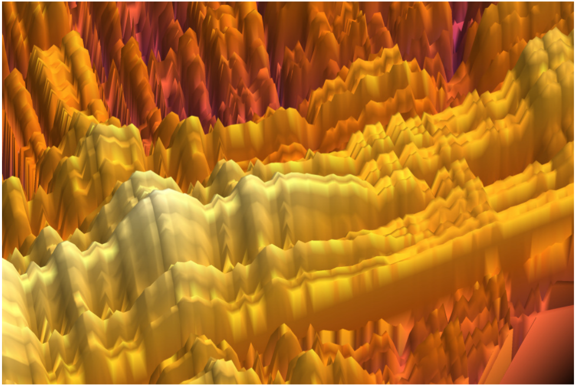


An introduction to core entropy

In honor of Mariusz Urbański, Bedlewo

Giulio Tiozzo
University of Toronto

Lecture I - Summary

1. What is... topological entropy?

Lecture I - Summary

1. What is... topological entropy?
2. A crash course in complex dynamics

Lecture I - Summary

1. What is... topological entropy?
2. A crash course in complex dynamics
3. The core entropy

Lecture I - Summary

1. What is... topological entropy?
2. A crash course in complex dynamics
3. The core entropy

Topological entropy

Let $f : X \to X$ be a continuous map of a compact Hausdorff space.

Topological entropy

Let $f : X \to X$ be a continuous map of a compact Hausdorff space.
For each cover $\mathcal{U}$ of X,

Topological entropy

Let $f : X \to X$ be a continuous map of a compact Hausdorff space.
For each cover $\mathcal{U}$ of X, let $N(\mathcal{U})$ the cardinality of a minimal subcover.

Topological entropy

Let $f : X \to X$ be a continuous map of a compact Hausdorff space.
For each cover $\mathcal{U}$ of X, let $N(\mathcal{U})$ the cardinality of a minimal subcover. Let $\mathcal{U} \vee \mathcal{V}$ be the common refinement of $\mathcal{U}, |V$.

Topological entropy

Let $f : X \to X$ be a continuous map of a compact Hausdorff space.
For each cover $\mathcal{U}$ of X, let $N(\mathcal{U})$ the cardinality of a minimal subcover. Let $\mathcal{U} \vee \mathcal{V}$ be the common refinement of $\mathcal{U}, |V$.

Then the topological entropy of f is

$$h_{top}(f) := \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))$$

Topological entropy of real interval maps

Let $f : I \to I$, continuous, piecewise monotone.

Topological entropy of real interval maps

Let $f : I \to I$, continuous, piecewise monotone. A lap of f is a maximal interval on which f is monotone.

Topological entropy of real interval maps

Let $f : I \to I$, continuous, piecewise monotone. A lap of f is a maximal interval on which f is monotone. The topological entropy of f also equals (Misiurewicz-Szlenk)

Topological entropy of real interval maps

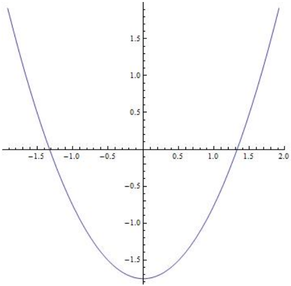
Let $f : I \to I$, continuous, piecewise monotone. A lap of f is a maximal interval on which f is monotone. The topological entropy of f also equals (Misiurewicz-Szlenk)

$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\mathrm{laps}(f^n)\}}{n}$$

Topological entropy of real interval maps

Let $f : I \to I$, continuous, piecewise monotone. A lap of f is a maximal interval on which f is monotone. The topological entropy of f also equals (Misiurewicz-Szlenk)

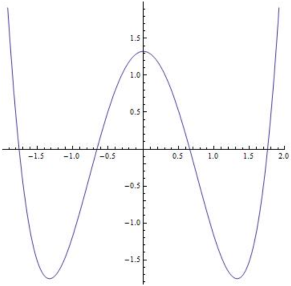
$$h_{top}(f, \mathbb{R}) = \lim_{n \to \infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

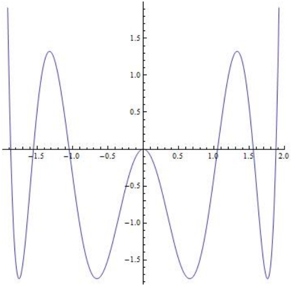
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

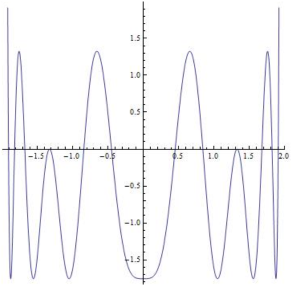
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

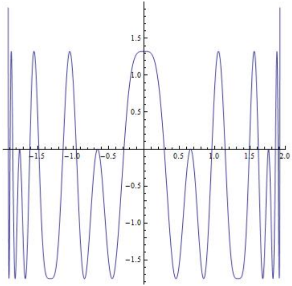
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

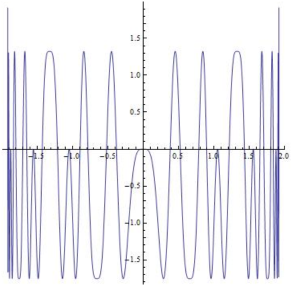
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

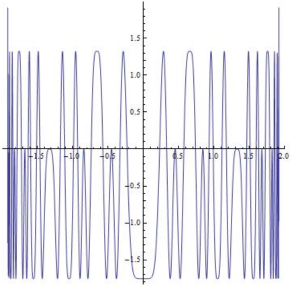
$$h_{top}(f, \mathbb{R}) = \lim_{n \to \infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

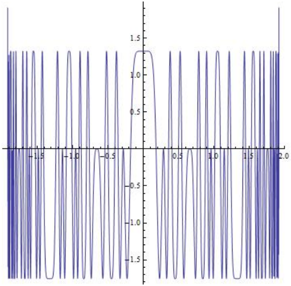
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

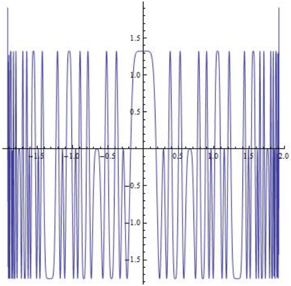
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

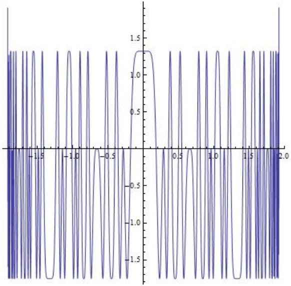
$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$



Topological entropy of real maps

Let $f : I \to I$, continuous, piecewise monotone.

$$h_{top}(f, \mathbb{R}) = \lim_{n\to\infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$

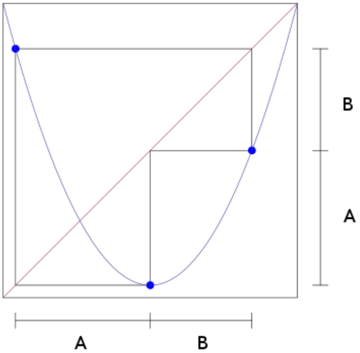


Example: the airplane map

$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite.

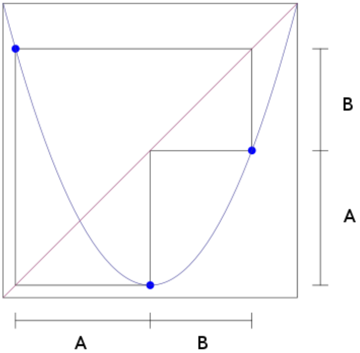
Example: the airplane map

$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite. Then the entropy is the logarithm of an algebraic number.



Example: the airplane map

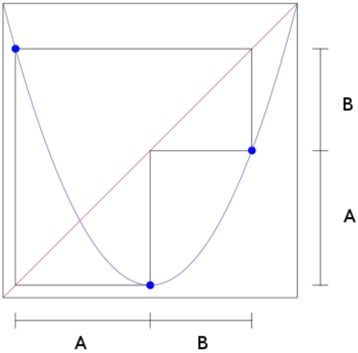
$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite. Then the entropy is the logarithm of an algebraic number.



$$\begin{aligned} A &\mapsto A \cup B \\ B &\mapsto A \end{aligned}$$

Example: the airplane map

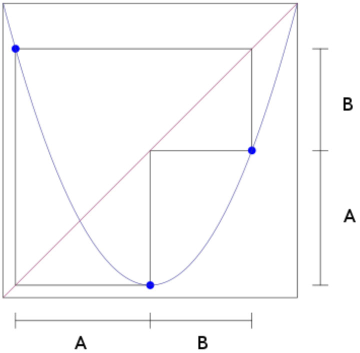
$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite. Then the entropy is the logarithm of an algebraic number.



$$\begin{array}{ccc} A & \mapsto & A \cup B \\ B & \mapsto & A \end{array} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Example: the airplane map

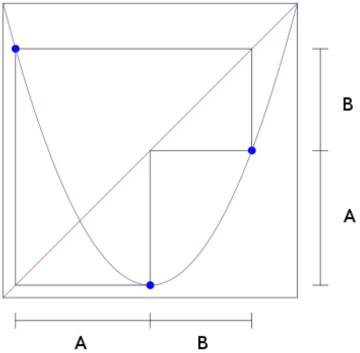
$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite. Then the entropy is the logarithm of an algebraic number.



$$\begin{array}{ccc} A & \mapsto & A \cup B \\ B & \mapsto & A \end{array} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda = \frac{\sqrt{5}+1}{2}$$

Example: the airplane map

$f : I \to I$ is postcritically finite if the forward orbits of the critical points of f are finite. Then the entropy is the logarithm of an algebraic number.



$$\begin{array}{ccc} A & \mapsto & A \cup B \\ B & \mapsto & A \end{array} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda = \frac{\sqrt{5}+1}{2} = e^{h_{top}(f_c, \mathbb{R})}$$

Topological entropy of real maps

$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log \#\{\text{laps}(f^n)\}}{n}$$

Consider the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

Topological entropy of real maps

$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log \#\{\mathrm{laps}(f^n)\}}{n}$$

Consider the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

How does entropy change with the parameter c?

The function $c \to h_{top}(f_c, \mathbb{R})$:

The function $c \to h_{top}(f_c, \mathbb{R})$:

- is continuous

The function $c \to h_{top}(f_c, \mathbb{R})$:

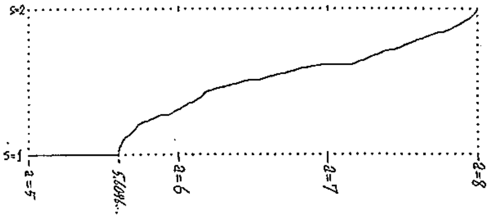
- is continuous and monotone (Milnor-Thurston, 1977).

The function $c \to h_{top}(f_c, \mathbb{R})$:

- is continuous and monotone (Milnor-Thurston, 1977).
- $0 \leq h_{top}(f_c, \mathbb{R}) \leq \log 2$.

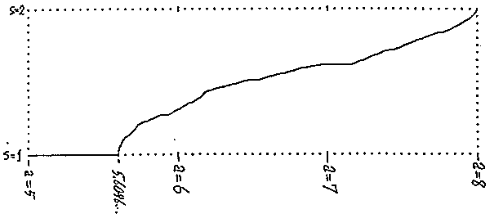
The function $c \to h_{top}(f_c, \mathbb{R})$:

- is continuous and monotone (Milnor-Thurston, 1977).
- $0 \leq h_{top}(f_c, \mathbb{R}) \leq \log 2$.



The function $c \to h_{top}(f_c, \mathbb{R})$:

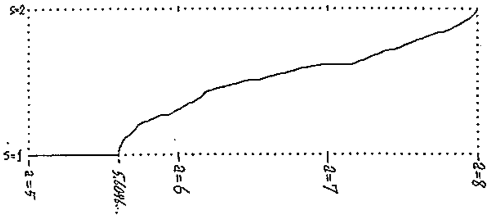
- is continuous and monotone (Milnor-Thurston, 1977).
- $0 \leq h_{top}(f_c, \mathbb{R}) \leq \log 2$.



[Picture is for $f_a(x) = ax(1-x)$.]

The function $c \to h_{top}(f_c, \mathbb{R})$:

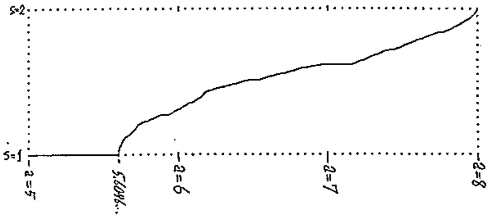
- is continuous and monotone (Milnor-Thurston, 1977).
- $0 \leq h_{top}(f_c, \mathbb{R}) \leq \log 2$.



Question : Can we extend this theory to complex polynomials?

The function $c \to h_{top}(f_c, \mathbb{R})$:

- is continuous and monotone (Milnor-Thurston, 1977).
- $0 \leq h_{top}(f_c, \mathbb{R}) \leq \log 2$.

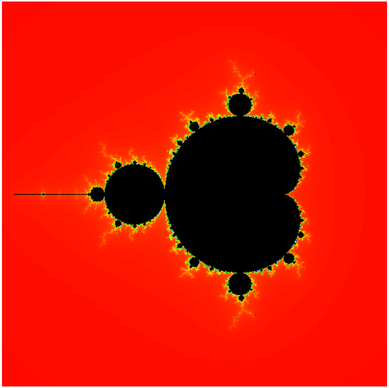


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$. (Lyubich 1980)

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \nrightarrow \infty\}$$



Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).

Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a

Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a hyperbolic component.

Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a hyperbolic component.
All polys in the same component H have the same period p, and every hyperbolic component is biholomorphic to the disk:

$$\lambda_H : H \to \mathbb{D}$$

$$\lambda_H(c) := (f_c^p)'(z)$$

where z is an attracting periodic point.

Hyperbolic components

A quadratic polynomial is <u>hyperbolic</u> if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a hyperbolic component.
All polys in the same component H have the same <u>period</u> p, and every hyperbolic component is biholomorphic to the disk:

$$\lambda_H : H \to \mathbb{D}$$

$$\lambda_H(c) := (f_c^p)'(z)$$

where z is an attracting periodic point. The parameter c for which $\lambda_H(c) = 0$ is called the <u>center</u> of H. The polynomial corresponding to the center has purely periodic critical orbit.

Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a hyperbolic component.
All polys in the same component H have the same period p, and every hyperbolic component is biholomorphic to the disk:

$$\lambda_H : H \to \mathbb{D}$$

$$\lambda_H(c) := (f_c^p)'(z)$$

where z is an attracting periodic point. The parameter c for which $\lambda_H(c) = 0$ is called the center of H. The polynomial corresponding to the center has purely periodic critical orbit.
The map λ_H extends to ∂H, and the parameter with $\lambda_H = 1$ is called a root of H. The polynomial corresponding to the root has a critical orbit which converges to a parabolic cycle.

Hyperbolic components

A quadratic polynomial is hyperbolic if its critical point converges to an attracting periodic cycle ($\neq \infty$).
The set of hyperbolic polynomials is open, and each connected component is called a hyperbolic component.
All polys in the same component H have the same period p, and every hyperbolic component is biholomorphic to the disk:

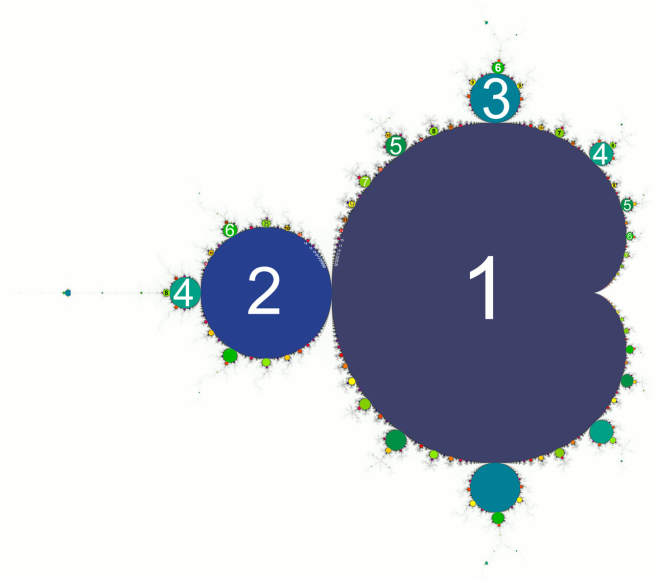
$$\lambda_H : H \to \mathbb{D}$$

$$\lambda_H(c) := (f_c^p)'(z)$$

where z is an attracting periodic point. The parameter c for which $\lambda_H(c) = 0$ is called the center of H. The polynomial corresponding to the center has purely periodic critical orbit.
The map λ_H extends to ∂H, and the parameter with $\lambda_H = 1$ is called a root of H. The polynomial corresponding to the root has a critical orbit which converges to a parabolic cycle.

Hyperbolic components

Each hyperbolic component has a period, and is biholomorphic to the disk.



External rays

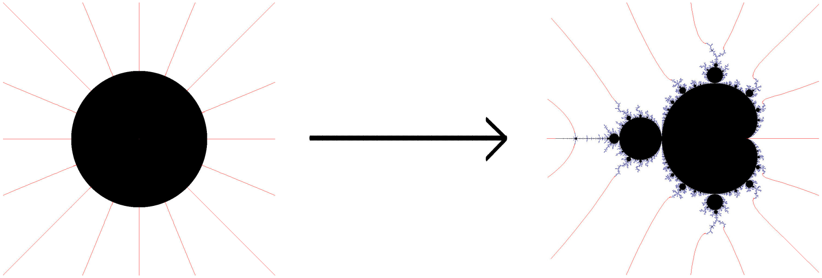
Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$



External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

The images of radial arcs in the disk are called external rays. Every angle $\theta \in \mathbb{R}/\mathbb{Z}$ determines an external ray

$$R(\theta) := \Phi_{\mathcal{M}}(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

The images of radial arcs in the disk are called external rays. Every angle $\theta \in \mathbb{R}/\mathbb{Z}$ determines an external ray

$$R(\theta) := \Phi_{\mathcal{M}}(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

An external ray $R(\theta)$ is said to land at x if

$$\lim_{\rho \to 1} \Phi_{\mathcal{M}}(\rho e^{2\pi i\theta}) = x$$

External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

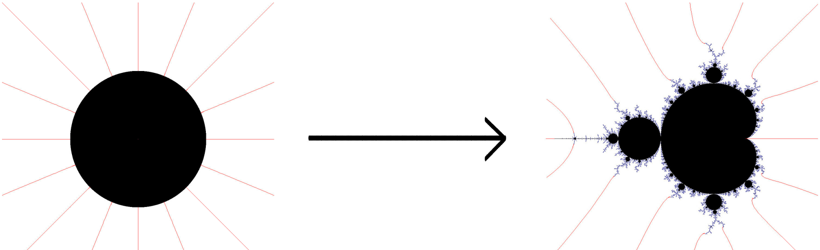
$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

The images of radial arcs in the disk are called external rays. Every angle $\theta \in \mathbb{R}/\mathbb{Z}$ determines an external ray

$$R(\theta) := \Phi_{\mathcal{M}}(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

An external ray $R(\theta)$ is said to land at x if

$$\lim_{\rho \to 1} \Phi_{\mathcal{M}}(\rho e^{2\pi i\theta}) = x$$



Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even,

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic (Misiurewicz parameter)

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic (Misiurewicz parameter)
- If $\theta = \frac{p}{q}$ with q odd,

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic (Misiurewicz parameter)
- If $\theta = \frac{p}{q}$ with q odd, then $R(\theta)$ lands at the root of some hyperbolic component;

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic (Misiurewicz parameter)
- If $\theta = \frac{p}{q}$ with q odd, then $R(\theta)$ lands at the root of some hyperbolic component; define f_θ as the center of such component.

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

- If $\theta = \frac{p}{q}$ with q even, then $R(\theta)$ lands at a parameter where the critical point is pre-periodic (Misiurewicz parameter)
- If $\theta = \frac{p}{q}$ with q odd, then $R(\theta)$ lands at the root of some hyperbolic component; define f_θ as the center of such component. Then f_θ is postcritically finite with purely periodic critical point.

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

Conjecture (Douady-Hubbard, MLC)

All rays land, and the boundary map $\mathbb{R}/\mathbb{Z} \to \partial\mathcal{M}$ is continuous.

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$*, then the external ray R*(θ) *lands and determines a postcritically finite quadratic polynomial* f_θ*.*

Conjecture (Douady-Hubbard, MLC)

All rays land, and the boundary map $\mathbb{R}/\mathbb{Z} \to \partial\mathcal{M}$ *is continuous.*

As a consequence, the Mandelbrot set is homeomorphic to a quotient of the closed disk

Rational rays land

Theorem (Douady-Hubbard, '84)

If $\theta \in \mathbb{Q}/\mathbb{Z}$, then the external ray $R(\theta)$ lands and determines a postcritically finite quadratic polynomial f_θ.

Conjecture (Douady-Hubbard, MLC)

All rays land, and the boundary map $\mathbb{R}/\mathbb{Z} \to \partial\mathcal{M}$ is continuous.

As a consequence, the Mandelbrot set is homeomorphic to a quotient of the closed disk (hence locally connected).

Julia sets

Let $f_c(z) = z^2 + c$. Then the filled Julia set of f_c is the set of points which do not escape to infinity under forward iteration:

$$K(f_c) := \{z \in \mathbb{C} \ : \ f_c^n(z) \text{ is bounded } \}$$

Julia sets

Let $f_c(z) = z^2 + c$. Then the filled Julia set of f_c is the set of points which do not escape to infinity under forward iteration:

$$K(f_c) := \{z \in \mathbb{C} \ : \ f_c^n(z) \text{ is bounded } \}$$

and the Julia set is its boundary:

$$J(f_c) := \partial K(f_c)$$

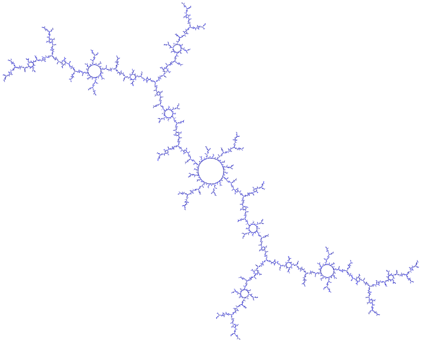
Julia sets

Let $f_c(z) = z^2 + c$. Then the filled Julia set of f_c is the set of points which do not escape to infinity under forward iteration:

$$K(f_c) := \{z \in \mathbb{C} \ : \ f_c^n(z) \text{ is bounded } \}$$

and the Julia set is its boundary:

$$J(f_c) := \partial K(f_c)$$



Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$.

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

$$\Phi_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f_c)$$

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

$$\Phi_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f_c)$$

One defines the external ray

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta}), \rho > 1\}$$

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

$$\Phi_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f_c)$$

One defines the external ray

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta}), \rho > 1\}$$

This map has the property

$$f_c(\Phi_c(z)) = \Phi_c(z^2) \qquad z \in \mathbb{C} \setminus \overline{\mathbb{D}}$$

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

$$\Phi_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f_c)$$

One defines the external ray

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta}), \rho > 1\}$$

This map has the property

$$f_c(\Phi_c(z)) = \Phi_c(z^2) \qquad z \in \mathbb{C} \setminus \overline{\mathbb{D}}$$

If $J(f_c)$ is locally connected, Φ_c extends to

$$\gamma_c := \partial\mathbb{D} = \mathbb{R}/\mathbb{Z} \to K(f_c)$$

Julia sets - II

The Fatou set is $\Omega(f_c) := \mathbb{C} \setminus J(f_c)$. If $J(f_c)$ is connected, one has the Riemann map

$$\Phi_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f_c)$$

One defines the external ray

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta}), \rho > 1\}$$

This map has the property

$$f_c(\Phi_c(z)) = \Phi_c(z^2) \qquad z \in \mathbb{C} \setminus \overline{\mathbb{D}}$$

If $J(f_c)$ is locally connected, Φ_c extends to

$$\gamma_c := \partial\mathbb{D} = \mathbb{R}/\mathbb{Z} \to K(f_c)$$

and we have

$$f_c(\gamma_c(\theta)) = \gamma_c(2\theta)$$

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

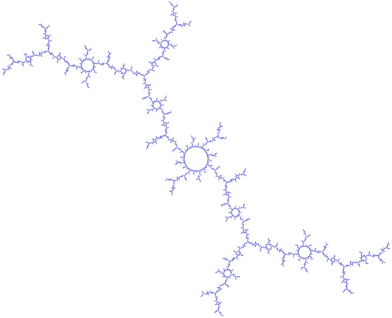
where $[x, y]$ is the regulated arc between x and y.

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

where $[x, y]$ is the regulated arc between x and y.

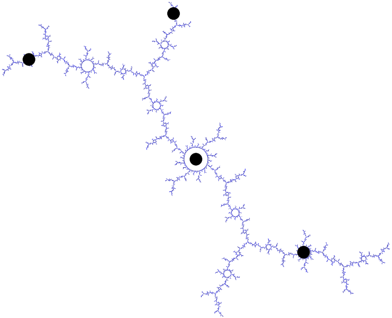


The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

where $[x, y]$ is the regulated arc between x and y.

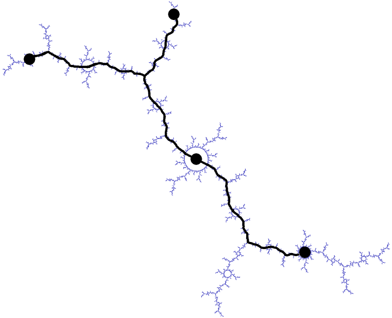


The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

where $[x, y]$ is the regulated arc between x and y.

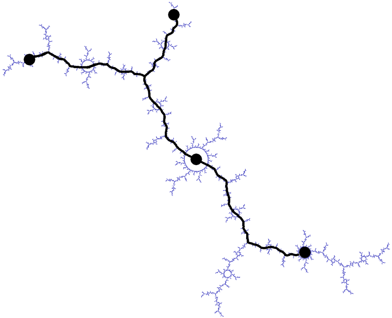


The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

It is a forward invariant, connected subset of the filled Julia set which contains the critical orbit.

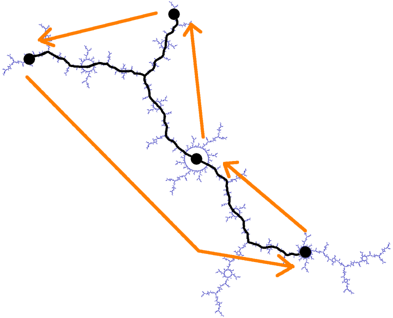


The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is

$$T_c := \bigcup_{m,n \geq 0} [f_c^m(0), f_c^n(0)]$$

It is a forward invariant, connected subset of the filled Julia set which contains the critical orbit. The map f_c acts on it.



The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected

The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected (e.g. a postcritically finite f).

The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected (e.g. a postcritically finite f). Then the core entropy of f

The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected (e.g. a postcritically finite f). Then the core entropy of f is the entropy of the restriction

The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected (e.g. a postcritically finite f). Then the core entropy of f is the entropy of the restriction

$$h(f) := h(f\,|_{T_f})$$

The core entropy

Definition (W. Thurston)

Let f be a polynomial whose Julia set is connected and locally connected (e.g. a postcritically finite f). Then the core entropy of f is the entropy of the restriction

$$h(f) := h(f\,|_{T_f})$$

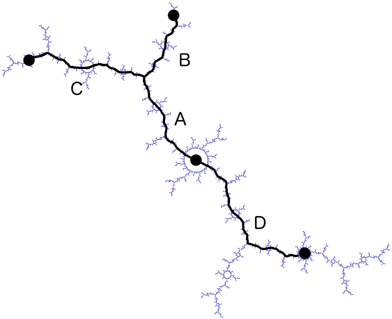
where T_f is the Hubbard tree of f.

The core entropy - example

$$h(f) := h(f\,|_{T_f})$$

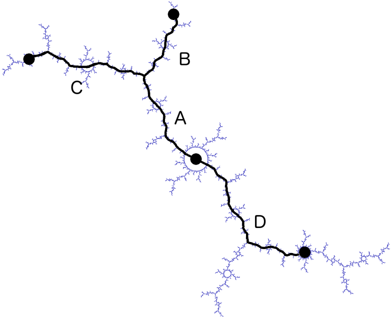
The core entropy - example

$$h(f) := h(f\,|_{T_f})$$



The core entropy - example

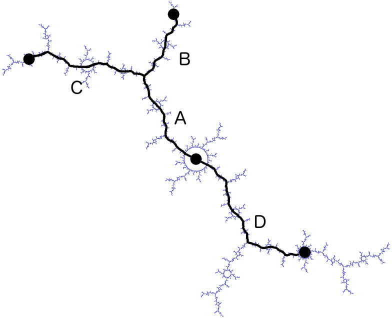
$$h(f) := h(f\,|_{T_f})$$



$A \to B$

The core entropy - example

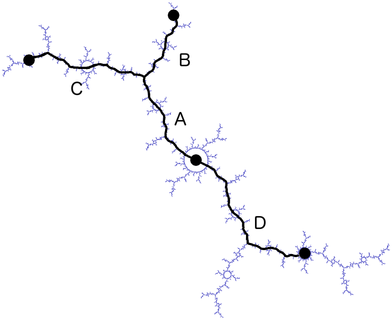
$$h(f) := h(f\,|_{T_f})$$



$A \to B$
$B \to C$

The core entropy - example

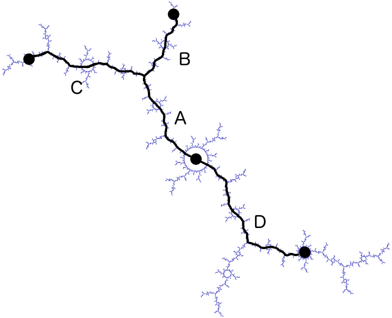
$$h(f) := h(f\,|_{T_f})$$



$A \to B$
$B \to C$
$C \to A \cup D$

The core entropy - example

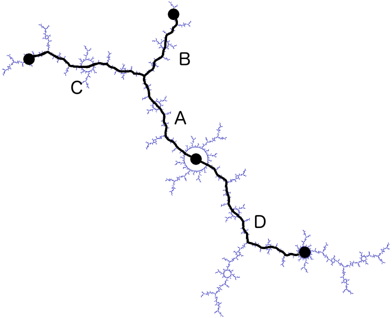
$$h(f) := h(f\,|_{T_f})$$



$A \to B$
$B \to C$
$C \to A \cup D$
$D \to A \cup B$

The core entropy - example

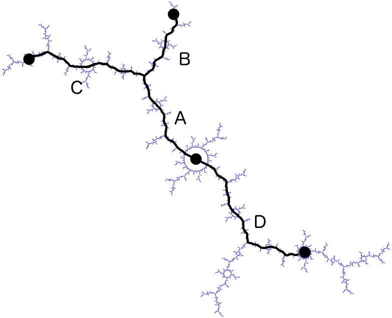
$$h(f) := h(f|_{T_f})$$



$$\begin{aligned} A &\to B \\ B &\to C \\ C &\to A \cup D \\ D &\to A \cup B \end{aligned} \qquad M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The core entropy - example

$$h(f) := h(f\,|_{T_f})$$



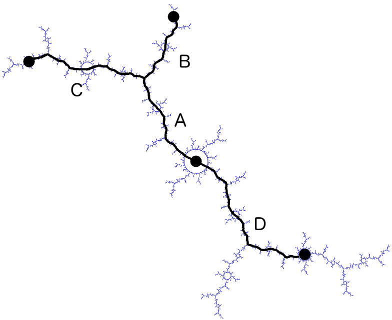
$$\begin{aligned} A &\to B \\ B &\to C \\ C &\to A \cup D \\ D &\to A \cup B \end{aligned}$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(M - xI) = \\ = -1 - 2x + x^4$$

The core entropy - example

$$h(f) := h(f|_{T_f})$$



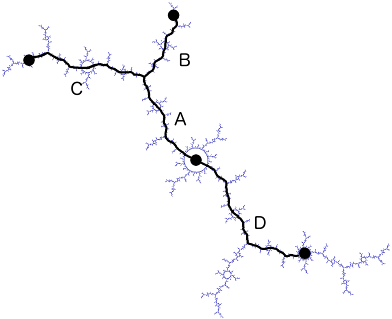
$$\begin{aligned} A &\to B \\ B &\to C \\ C &\to A \cup D \\ D &\to A \cup B \end{aligned}$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(M - xI) =$$
$$= -1 - 2x + x^4$$
$$\lambda \approx 1.39534$$

The core entropy - example

$$h(f) := h(f\,|_{T_f})$$



$$A \to B$$
$$B \to C$$
$$C \to A \cup D$$
$$D \to A \cup B$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(M - xI) =$$
$$= -1 - 2x + x^4$$
$$\lambda \approx 1.39534$$
$$h \approx \log 1.39534$$

The core entropy

Let $\theta \in \mathbb{Q}/\mathbb{Z}$. Then the external ray at angle θ lands, and determines a postcritically finite quadratic polynomial f_θ, with Hubbard tree T_θ.

The core entropy

Let $\theta \in \mathbb{Q}/\mathbb{Z}$. Then the external ray at angle θ lands, and determines a postcritically finite quadratic polynomial f_θ, with Hubbard tree T_θ.

Definition (W. Thurston)

The core entropy of f_θ is

$$h(\theta) := h(f_\theta \mid_{T_\theta})$$

The core entropy

Let $\theta \in \mathbb{Q}/\mathbb{Z}$. Then the external ray at angle θ lands, and determines a postcritically finite quadratic polynomial f_θ, with Hubbard tree T_θ.

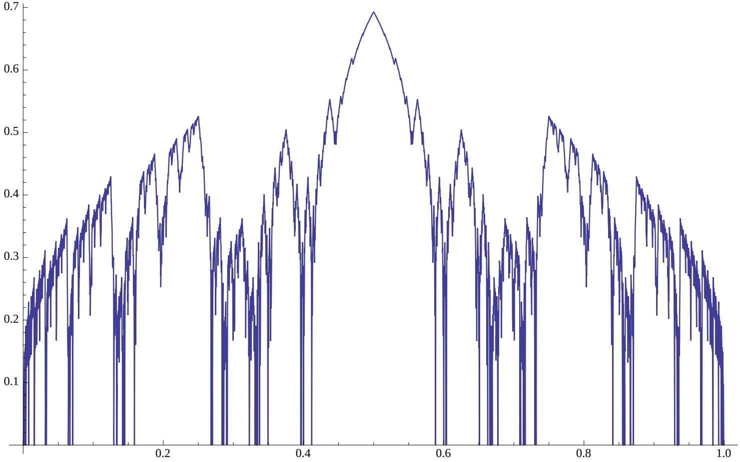
Definition (W. Thurston)

The core entropy of f_θ is

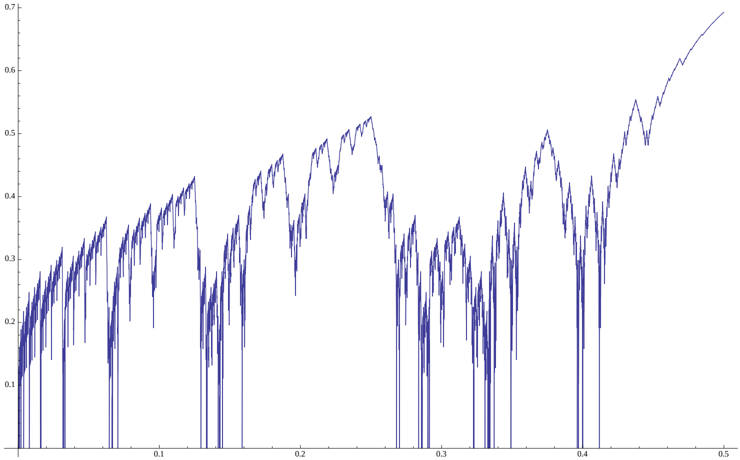
$$h(\theta) := h(f_\theta \mid_{T_\theta})$$

Question: How does $h(\theta)$ vary with the parameter θ?

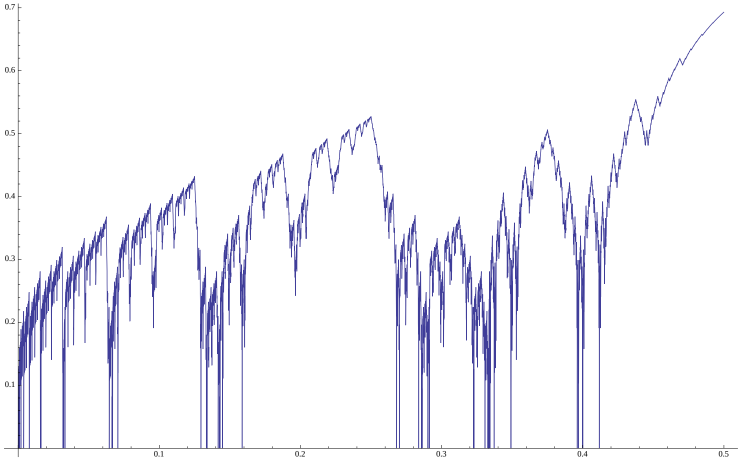
Core entropy as a function of external angle (W. Thurston)



Core entropy as a function of external angle (W. Thurston)



Core entropy as a function of external angle (W. Thurston)



Question Can you see the Mandelbrot set in this picture?

Monotonicity of entropy

Observation.

Monotonicity of entropy

Observation.
If $R_M(\theta_1)$ and $R_M(\theta_2)$ land together, then $h(\theta_1) = h(\theta_2)$.

Monotonicity of entropy

<u>Observation.</u>
If $R_M(\theta_1)$ and $R_M(\theta_2)$ land together, then $h(\theta_1) = h(\theta_2)$.

Monotonicity still holds along veins.

Monotonicity of entropy

Observation.
If $R_M(\theta_1)$ and $R_M(\theta_2)$ land together, then $h(\theta_1) = h(\theta_2)$.

Monotonicity still holds along veins.

Let us take two rays θ_1 landing at c_1 and θ_2 landing at c_2.

Monotonicity of entropy

Observation.
If $R_M(\theta_1)$ and $R_M(\theta_2)$ land together, then $h(\theta_1) = h(\theta_2)$.

Monotonicity still holds along veins.

Let us take two rays θ_1 landing at c_1 and θ_2 landing at c_2.
Then we define $\theta_1 <_M \theta_2$ if c_1 lies on the arc $[0, c_2]$.

Monotonicity of entropy

<u>Observation.</u>
If $R_M(\theta_1)$ and $R_M(\theta_2)$ land together, then $h(\theta_1) = h(\theta_2)$.

Monotonicity still holds along veins.

Let us take two rays θ_1 landing at c_1 and θ_2 landing at c_2.
Then we define $\theta_1 <_M \theta_2$ if c_1 lies on the arc $[0, c_2]$.

Theorem (Li Tao; Penrose; Tan Lei; Zeng Jinsong)

If $\theta_1 <_M \theta_2$, then

$$h(\theta_1) \leq h(\theta_2)$$

Core entropy and biaccessibility

The core entropy is also proportional to the dimension of the set of **biaccessible angles** (Zakeri, Smirnov, Zdunik, Bruin-Schleicher ...)

Core entropy and biaccessibility

The core entropy is also proportional to the dimension of the set of **biaccessible angles** (Zakeri, Smirnov, Zdunik, Bruin-Schleicher ...)

Definition

θ is biaccessible if $\exists \eta \neq \theta$ s.t. $R(\theta)$ and $R(\eta)$ land at the same point.

Core entropy and biaccessibility

The core entropy is also proportional to the dimension of the set of **biaccessible angles** (Zakeri, Smirnov, Zdunik, Bruin-Schleicher ...)

Definition

θ is biaccessible if $\exists \eta \neq \theta$ s.t. $R(\theta)$ and $R(\eta)$ land at the same point.

Let us denote the set of biaccessible angles as

$$B_c := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \theta \text{ is biaccessible } \}$$

Core entropy and biaccessibility

The core entropy is also proportional to the dimension of the set of **biaccessible angles** (Zakeri, Smirnov, Zdunik, Bruin-Schleicher ...)

Definition

θ is biaccessible if $\exists \eta \neq \theta$ s.t. $R(\theta)$ and $R(\eta)$ land at the same point.

Let us denote the set of biaccessible angles as

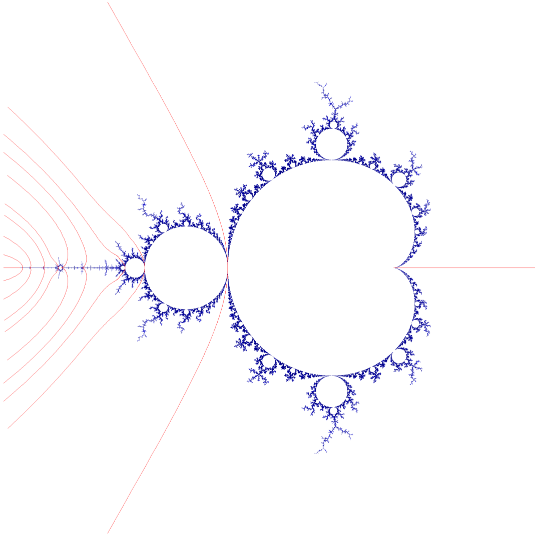
$$B_c := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \theta \text{ is biaccessible } \}$$

Theorem (T., Bruin-Schleicher)

If the Hubbard tree of f_c is topologically finite, then

$$\text{H. dim } B_c = \frac{h(f_c)}{\log 2}$$

Rays landing on the real slice of the Mandelbrot set



Harmonic measure

Given a subset A of $\partial\mathcal{M}$, the harmonic measure $\nu_{\mathcal{M}}$ is the probability that a random ray lands on A:

$$\nu_{\mathcal{M}}(A) := \mathrm{Leb}(\{\theta \in S^1 \ : \ R(\theta) \text{ lands on } A\})$$

Harmonic measure

Given a subset A of $\partial\mathcal{M}$, the harmonic measure $\nu_{\mathcal{M}}$ is the probability that a random ray lands on A:

$$\nu_{\mathcal{M}}(A) := \mathrm{Leb}(\{\theta \in S^1 \ : \ R(\theta) \text{ lands on } A\})$$

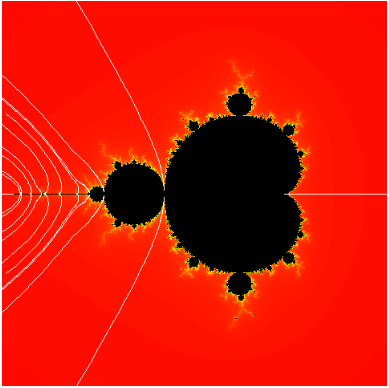
For instance, take $A = \mathcal{M} \cap \mathbb{R}$ the real section of the Mandelbrot set.

Harmonic measure

Given a subset A of $\partial\mathcal{M}$, the harmonic measure $\nu_{\mathcal{M}}$ is the probability that a random ray lands on A:

$$\nu_{\mathcal{M}}(A) := \mathrm{Leb}(\{\theta \in S^1 \ : \ R(\theta) \text{ lands on } A\})$$

For instance, take $A = \mathcal{M} \cap \mathbb{R}$ the real section of the Mandelbrot set. How common is it for a ray to land on the real axis?



Real section of the Mandelbrot set

Theorem (Zakeri, 2000)

The harmonic measure of the real axis is 0.

Real section of the Mandelbrot set

Theorem (Zakeri, 2000)

The harmonic measure of the real axis is 0. However,

Real section of the Mandelbrot set

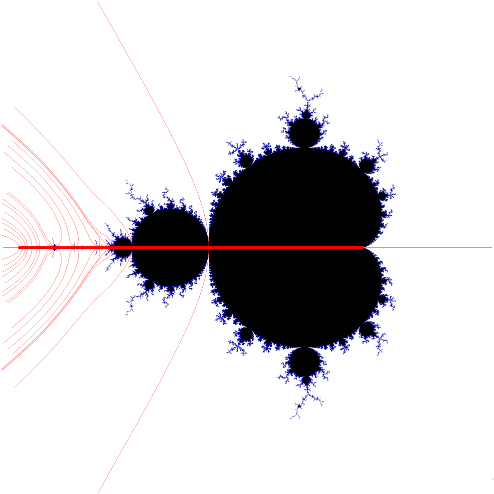
Theorem (Zakeri, 2000)

The harmonic measure of the real axis is 0. However, the Hausdorff dimension of the set of rays landing on the real axis is 1.

Real section of the Mandelbrot set

Theorem (Zakeri, 2000)

The harmonic measure of the real axis is 0. However, the Hausdorff dimension of the set of rays landing on the real axis is 1.



Sectioning $\mathcal{M}$

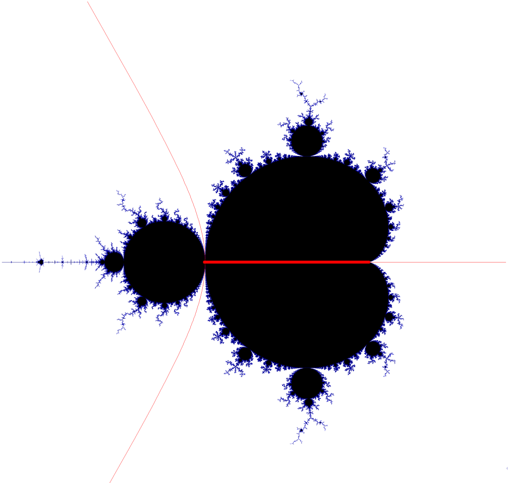
Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

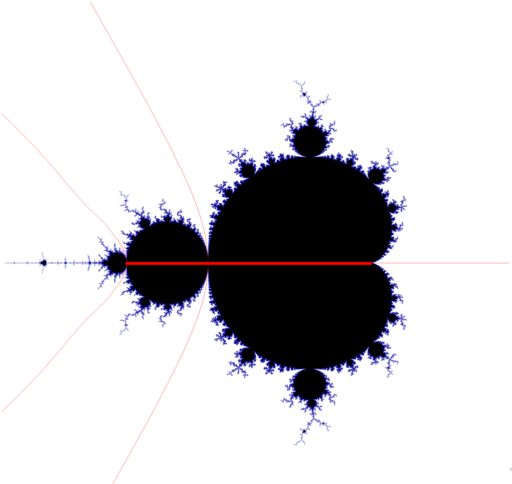
$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$



Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

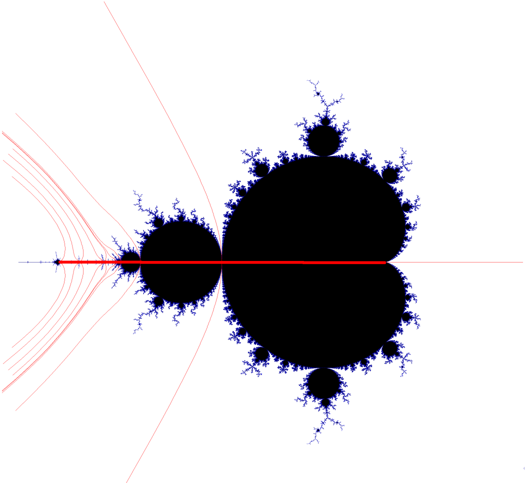
$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$



Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

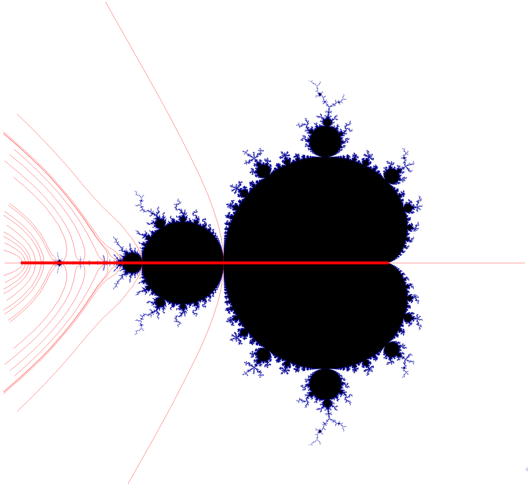
$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$



Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

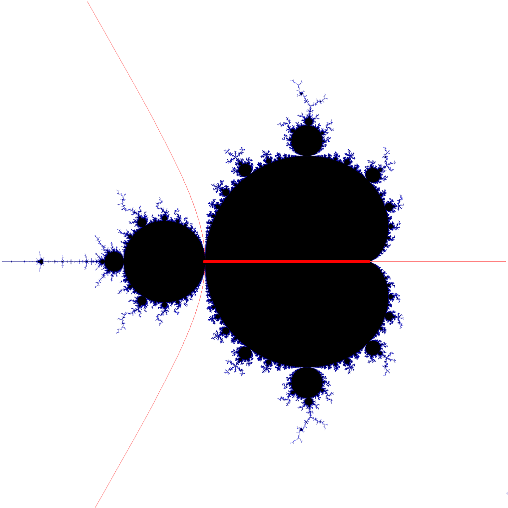


$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.

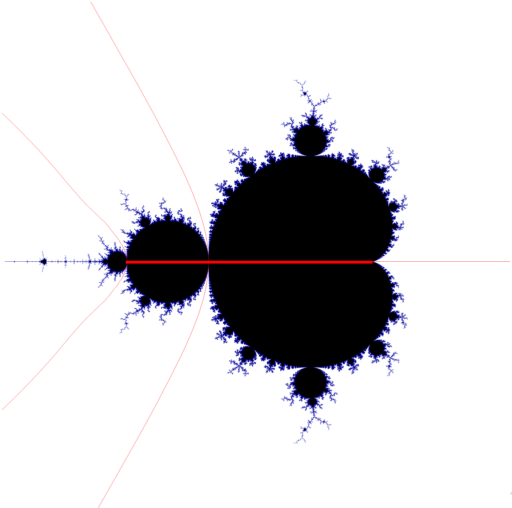


$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.

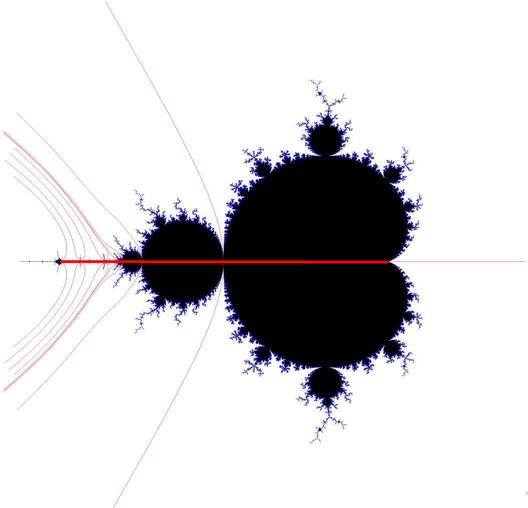


$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.

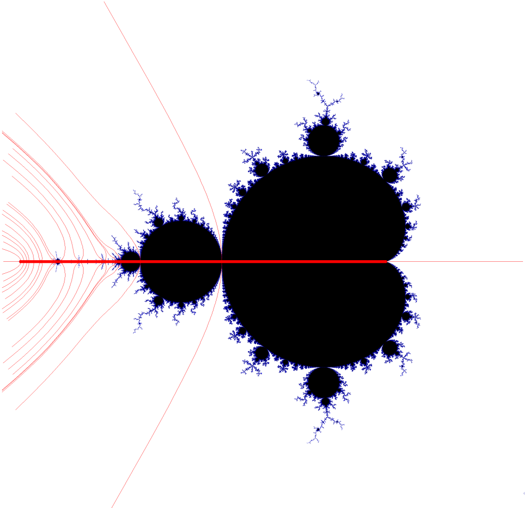


$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.



Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$. *Then*

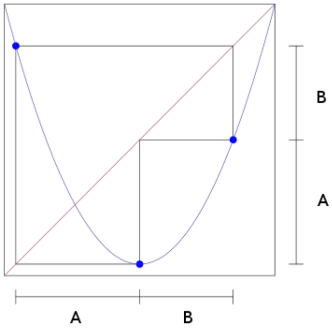
$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

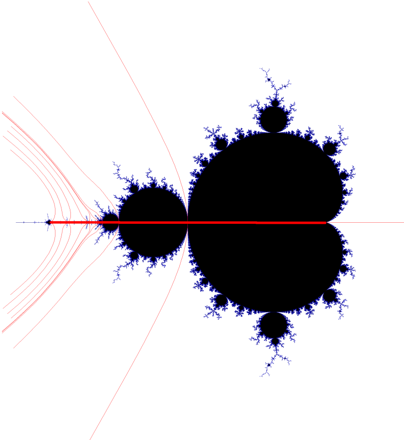
Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$*. Then*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$





Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$. *Then*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$. *Then*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

- It relates dynamical properties of a particular map to the geometry of parameter space near the chosen parameter.

Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$*. Then*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

- It relates dynamical properties of a particular map to the geometry of parameter space near the chosen parameter.
- Entropy formula: relates dimension, entropy and Lyapunov exponent (Manning, Bowen, Ledrappier, Young, ...).

Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$. Then

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

- It relates dynamical properties of a particular map to the geometry of parameter space near the chosen parameter.
- Entropy formula: relates dimension, entropy and Lyapunov exponent (Manning, Bowen, Ledrappier, Young, ...).
- The proof is combinatorial.

Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$. Then

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

- It relates dynamical properties of a particular map to the geometry of parameter space near the chosen parameter.
- Entropy formula: relates dimension, entropy and Lyapunov exponent (Manning, Bowen, Ledrappier, Young, ...).
- The proof is combinatorial.
- It does not depend on MLC.

Entropy formula, real case

Theorem (T.)

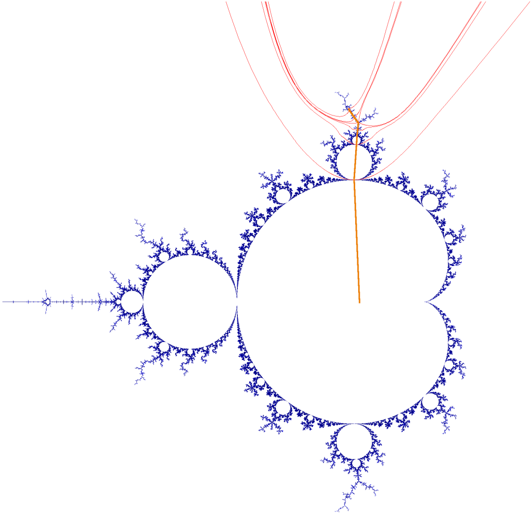
Let $c \in [-2, 1/4]$. *Then*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } P_c$$

- It relates dynamical properties of a particular map to the geometry of parameter space near the chosen parameter.
- Entropy formula: relates dimension, entropy and Lyapunov exponent (Manning, Bowen, Ledrappier, Young, ...).
- The proof is combinatorial.
- It does not depend on MLC.
- It can be generalized to non-real veins.

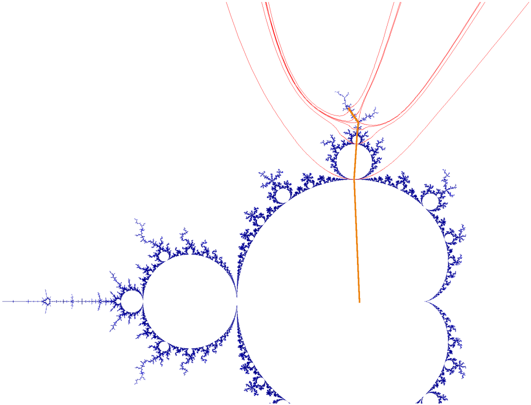
Entropy formula along complex veins

A vein is an embedded arc in the Mandelbrot set.



Entropy formula, complex case

A vein is an embedded arc in the Mandelbrot set.



Given a parameter c along a vein, we can look at the set P_c of parameter rays which land on the vein between 0 and c.

Entropy formula along complex veins

Theorem (T.; Jung)

Let v be a vein in the Mandelbrot set, and let $c \in v$.

Entropy formula along complex veins

Theorem (T.; Jung)

Let v be a vein in the Mandelbrot set, and let $c \in v$. Then

$$\frac{h(f_c)}{\log 2} =$$

Entropy formula along complex veins

Theorem (T.; Jung)

Let v be a vein in the Mandelbrot set, and let $c \in v$. Then

$$\frac{h(f_c)}{\log 2} = \text{H.dim } B_c$$

Entropy formula along complex veins

Theorem (T.; Jung)

Let v be a vein in the Mandelbrot set, and let $c \in v$. Then

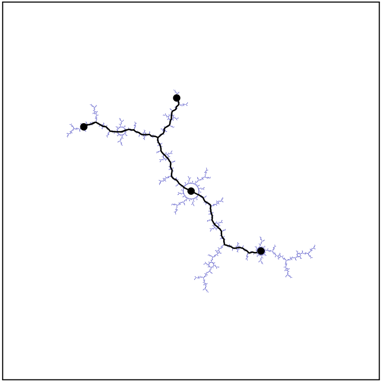
$$\frac{h(f_c)}{\log 2} = \text{H.dim } B_c = \text{H.dim } P_c$$

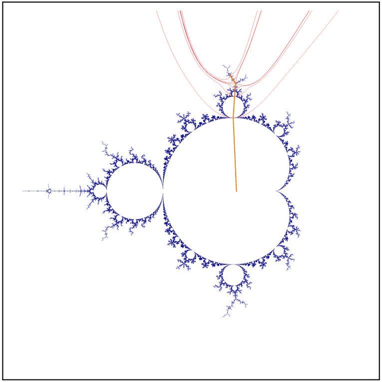
Entropy formula along complex veins

Theorem (T.; Jung)

Let v be a vein in the Mandelbrot set, and let $c \in v$. Then

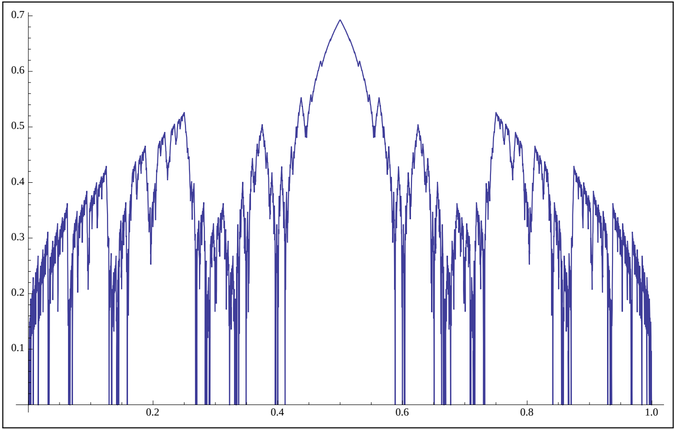
$$\frac{h(f_c)}{\log 2} = \text{H.dim } B_c = \text{H.dim } P_c$$





The core entropy as a function of external angle

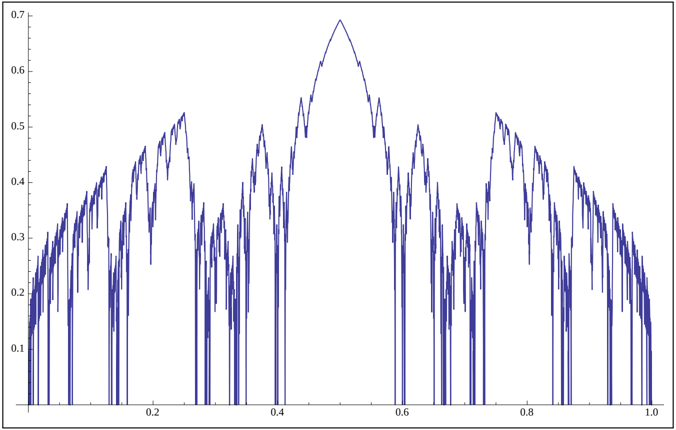
Question (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of θ?



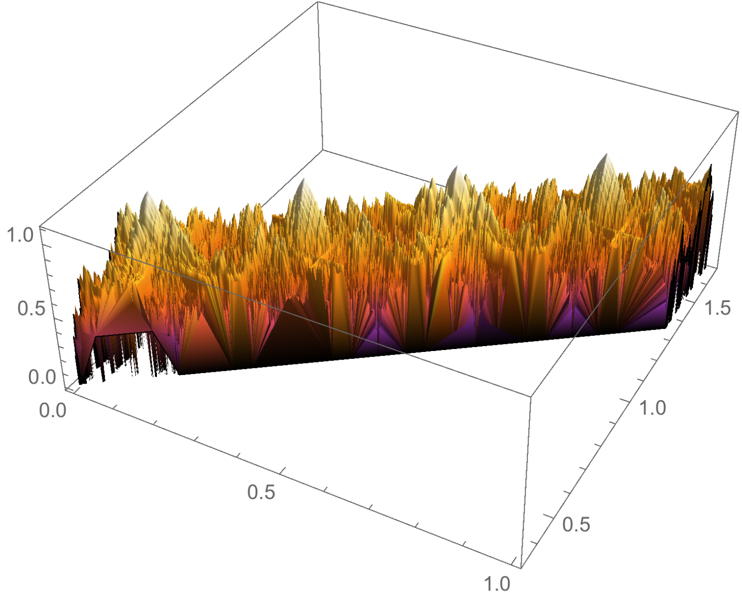
The Main Theorem: Continuity

Theorem (T.)

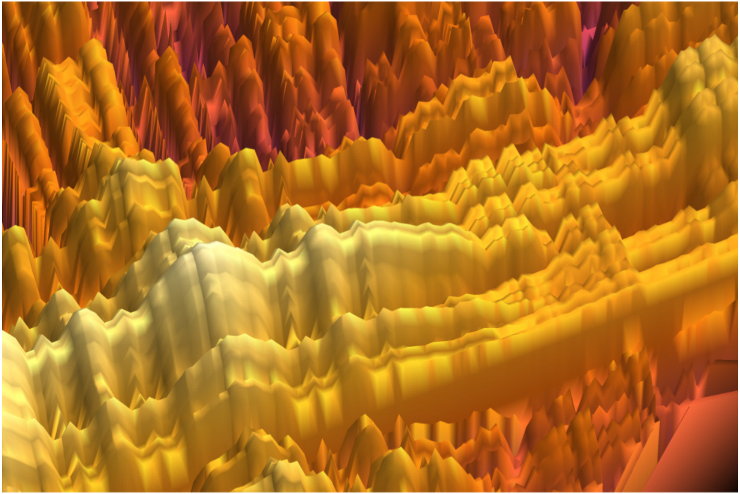
The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.



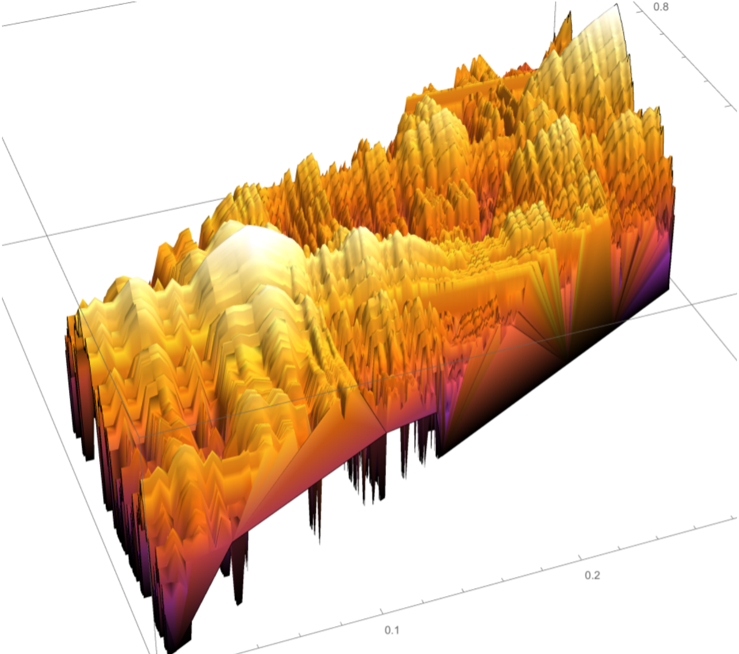
The core entropy for cubic polynomials



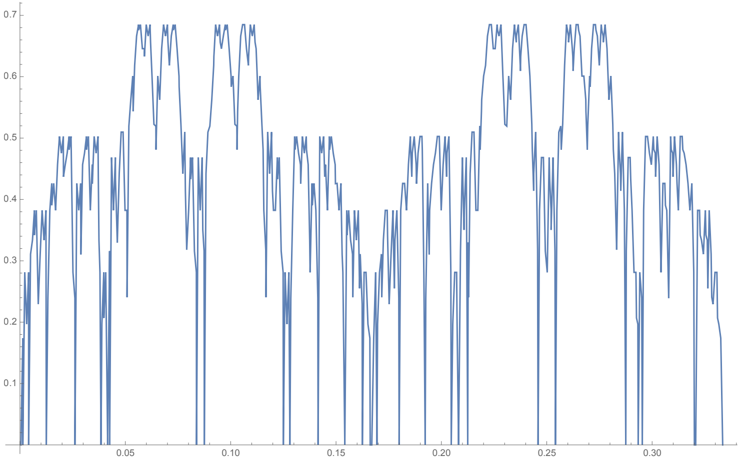
The core entropy for cubic polynomials



The core entropy for cubic polynomials

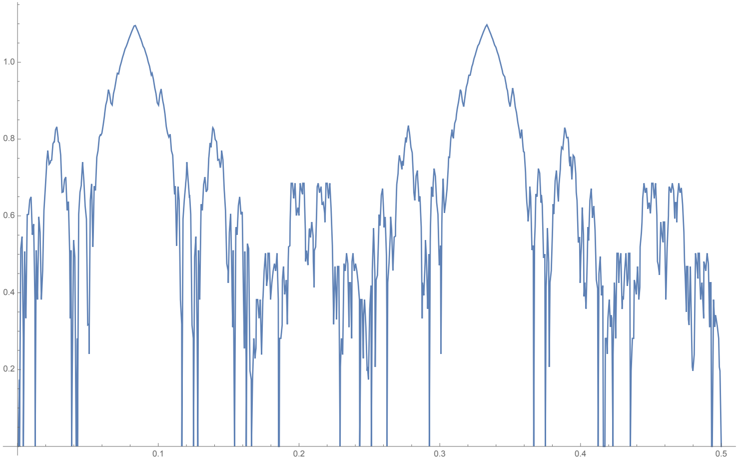


The unicritical slice



$$f(z) = z^3 + c$$

The symmetric slice



$$f(z) = z^3 + cz$$

Continuity in higher degree, combinatorial version

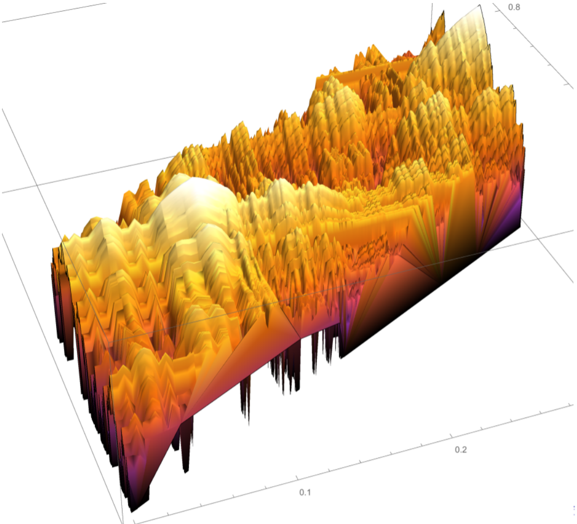
Theorem (T. - Yan Gao)

Fix $d \geq 2$. Then the core entropy extends to a continuous function on the space $\mathrm{PM}(d)$ of primitive majors.

Continuity in higher degree, combinatorial version

Theorem (T. - Yan Gao)

Fix $d \geq 2$. Then the core entropy extends to a continuous function on the space $\mathrm{PM}(d)$ of primitive majors.



Continuity in higher degree, analytic version

Define $\mathcal{P}_d$ as the space of monic, centered polynomials of degree d.

Continuity in higher degree, analytic version

Define $\mathcal{P}_d$ as the space of monic, centered polynomials of degree d. One says $f_n \to f$ if the coefficients of f_n converge to the coefficients of f.

Continuity in higher degree, analytic version

Define $\mathcal{P}_d$ as the space of monic, centered polynomials of degree d. One says $f_n \to f$ if the coefficients of f_n converge to the coefficients of f.

Theorem (T. - Yan Gao)

Let $d \geq 2$. Then the core entropy is a continuous function on the space of monic, centered, postcritically finite polynomials of degree d.

The end

Dziękuję!
Thank you!