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NON-PRODUCTIVELY LINDELÖF SPACES AND SMALL CARDINALS

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Abstract. A space is productively Lindelöf if its product with any Lindelöf space is Lindelöf. In this paper we concentrate on finding conditions for a Lindelöf space to not be productively Lindelöf, using small cardinals.

1. Introduction

A space is *productively Lindelöf* if its product with any Lindelöf space is Lindelöf. The question of which Lindelöf spaces are productively Lindelöf has a long history. Many references can be found in [16]. Here we just mention Michael [13], Przymusiński [15], and Barr et al [6].

The related question of what are the consequences of being productively Lindelöf was investigated by Alster [2], Aurichi and Tall [5], and by Tall [16]. In this paper we concentrate on finding conditions for a Lindelöf space to not be productively Lindelöf, using *small cardinals*, i.e. cardinals related to the structure of ${}^{\omega}\omega$. For these, see [9].

2. Notation and basic results

Several times we will deal with various topologies on the same set. To avoid confusion, we will say that a subset is τ-P, if it has the property P when using the topology τ.

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The following definition and theorem can be found in [10].

Definition 2.1. We say that a Tychonoff space X is Lindelöf at infinity if X^* $(= \beta X \setminus X)$ is a Lindelöf space.

Theorem 2.2 (Henriksen, Isbell [10])**.** *Let X be a Tychonoff Lindelöf space. Then X is Lindelöf at infinity if, and only if, for every compact $K \subset X$, there is a compact $K' \supset K$ such that K' has countable character in X.*

Note 2.3. Arhangel′skiĭ([4]) has studied such spaces under the name "spaces of countable type".

The following is the main result of this work. Most of the other results are consequences of it, or are proved using similar techniques.

Theorem 2.4. *Let (X, τ) be a regular Lindelöf space, Lindelöf at infinity. Suppose there is a $Y \subset X$ and a topology ρ over Y such that:*

(i) $\tau \restriction Y \subset \rho$;
(ii) (Y, ρ) *is regular and non-Lindelöf;*
(iii) *any $K \subset X$ that is τ-compact is such that $K \cap Y$ is ρ-Lindelöf;*
(iv) *for every $y \in Y$, there is a τ-(σ-compact) K (not necessarily contained in Y) such that $y \in K \cap Y \in \rho$.*

Then, X is not productively Lindelöf.

Proof. Let Z be the space $X^* \cup Y$ (X considered as (X, τ)) with the following topology. For the points in X^*, the open neighborhoods are of the form $V \setminus K$, where V is an usual open neighborhood of x in βX and $K \subset X$ is a τ-(σ-compact) set such that $K \cap Y \in \rho$. For the points in Y, we use the open sets in ρ. Note that Z is Lindelöf, since X^* is Lindelöf and for any open set V, such that $V \supset X^*$, $Y \setminus V$ is a σ-compact set in the topology τ, therefore, a Lindelöf space in the topology ρ.

To see that Z is regular, pick $z \in Z$. If $z \in X^*$, z has a closed neighborhood system because βX is regular. If $z \in Y$, there is a τ-(σ-compact) K such that $z \in K$ and $K \cap Y \in \rho$. Since (Y, ρ) is regular, there is an ρ-open set U such that $y \in U \subset \overline{U}^Y \subset K \cap Y$. Note that $\overline{U}^Y = \overline{U}^Z$ because $Z \setminus K$ is an open set that contains X^*.

Now let us prove that $X \times Z$ is non-Lindelöf. Consider $D = \{(y, y) : y \in Y\}$. Note that D is a closed subspace of $X \times Z$ and that it is homeomorphic to (Y, ρ), thus it is non-Lindelöf. □

Corollary 2.5. *Let X be a Lindelöf regular space, Lindelöf at infinity. If there is an uncountable set $Y \subset X$ such that, for every compact K, $K \cap Y$ is countable, then X is not productively Lindelöf.*

PROOF. Simply define ρ as the discrete topology over Y and apply the previous result. $\square$

Michael [13] proved this for the case of X being a Bernstein subset of the real line.

3. The $\mathcal{F}$-topology

In this section we develop a technique that can be used to construct Michael spaces (*i*.e. regular Lindelöf spaces such that their products with the space of the irrationals numbers are not Lindelöf) using Theorem 2.4. This technique can be used to show the existence of a Michael space under many already known hypotheses.

Definition 3.1. Let $\mathcal{F} \subset \omega^\omega$. We call the $\mathcal{F}$-topology over ω^ω the smallest topology that contains the usual one and is such that each $K_f = \{g \in \omega^\omega : g \leq f\}$ for $f \in \mathcal{F}$ is an open set. We will use this terminology also for subspaces $Y \subset \omega^\omega$ with the topology induced by a $\mathcal{F}$-topology (with $\mathcal{F}$ not necessarily contained in Y).

Proposition 3.2. *If we put $X = Y = \omega^\omega$ in Theorem 2.4 and we require that ρ is a $\mathcal{F}$-topology, then the condition (iv) of the theorem is equivalent to $\mathcal{F}$ being a dominating family.*

PROOF. Suppose $\mathcal{F}$ is dominating. Note that each K_f for $f \in \mathcal{F}$ is τ-compact, ρ-open, and $(K_f)_{f \in \mathcal{F}}$ is a covering for ω^ω.

Now suppose $\mathcal{F}$ is not dominating and let $h \in \omega^\omega \setminus \bigcup_{f \in \mathcal{F}} K_f$. Note that any τ-compact that is in ρ is a subset of some K_f. Therefore, there is no τ-(σ-compact) that contains h and is in ρ. $\square$

Proposition 3.3. *If we put $X = Y = \omega^\omega$ in Theorem 2.4 and we require that ρ is a $\mathcal{F}$-family, then there is a topology ρ that also satisfies the property that every τ-compact is ρ-(second countable) if, and only if, $\mathfrak{d} = \omega_1$.*

PROOF. Suppose $\mathfrak{d} = \omega_1$. Let $\mathcal{F}$ be a scale in ω^ω and ρ the $\mathcal{F}$-topology. Note that ρ satisfies the conditions of the Proposition 2.4. Thus we only have to prove that every τ-compact is ρ-second countable. Since every τ-compact subspace is contained in some K_f, we only have to show that every K_f is ρ-second countable. Let $\mathcal{G} = \{g \in \mathcal{F} : g \leq f\}$. Note that the sets of the form

$$\{g \in \omega^\omega : g \upharpoonright n = s \text{ and } g(m) \leq t(m) \text{ for } m \geq n\}$$

for $n \in \omega$, $s \in \omega^{<\omega}$ such that there is $h \in K_f$ such that $s = h \upharpoonright n$ and $t \in G$ form a ρ-countable base for K_f.

Suppose that every τ-compact is ρ-(second countable). First we will prove that $\mathfrak{b} = \omega_1$. Let $(f_\xi)_{\xi<\omega_1}$ be a family such that $f_\alpha \leq^* f_\beta$ if $\alpha \leq \beta$. Suppose that $(f_\xi)_{\xi<\omega_1}$ is bounded by f. Note that the set $A = \{g : g \leq^* f\}$ is τ-(σ-compact) and therefore is ρ-(second countable). Since $\{f_\xi : \xi < \omega_1\} \subset A$, there is an f_η that is a complete accumulation point of $\{f_\xi : \xi < \omega_1\}$. But K_{f_η} is a ρ-neighborhood of f_η that separates it from every f_ξ with $\xi > \eta$.

Now if we prove that $\mathfrak{b} = \mathfrak{d}$ we are done. Suppose $\omega_1 = \mathfrak{b} < \mathfrak{d}$ and and suppose $\mathcal{F}$ is dominating (we can do this by the previous result). Let $\{f_\xi : \xi < \omega_1\} \subset \mathcal{F}$ be such that $f_\alpha \leq^* f_\beta$ if $\alpha \leq \beta$ and $\{n \in \omega : f_\alpha(n) = f_\beta(n)\}$ is finite if $\alpha < \beta$.

Since $\mathfrak{d} > \omega_1$, there is a $g \in \mathcal{F}$ such that $g \not\leq^* f_\xi$ for all $\xi < \omega_1$. Thus, for every $\xi < \omega_1$, $\{n \in \omega : g(n) > f_\xi(n)\}$ is infinite. For each $\xi < \omega_1$, define $g_\xi \in \omega^\omega$ as

$$g_\xi(n) = \begin{cases} f_\xi(n) & \text{if } f_\xi(n) \leq g(n) \\ g(n) & \text{otherwise} \end{cases}$$

Note that every $g_\xi \in K_g$. Therefore, there is a g_α that is a ρ-(complete accumulation point) for $\{g_\xi : \xi < \omega_1\}$ since K_g is ρ-(second countable). Note that $g_\alpha \in K_{f_\alpha}$. By the way we chose the f_ξ's there is no $g_\beta \in K_{f_\alpha}$ with $\beta > \alpha$, contradicting the fact that g_α is a complete accumulation point. □

Lawrence's result [12] can be easily obtained after Corollary 2.5: assuming $\mathfrak{b} = \omega_1$ there is an uncountable $Y \subset \omega^\omega$ such that, for every compact $K \subset \omega^\omega$, $K \cap Y$ is countable (simply define Y as the set of a well-ordered and unbounded sequence of length ω_1). We can also prove this result using an $\mathcal{F}$-topology:

Proposition 3.4. *If $\mathfrak{b} = \omega_1$, there is a $Y \subset \omega^\omega$ and an $\mathcal{F} \subset Y$ such that Y with the $\mathcal{F}$-topology witnesses Theorem 2.4.*

Proof. Let $\mathcal{F} = \{f_\xi : \xi < \omega_1\}$ be an unbounded and well ordered sequence in ω^ω. Let $Y = \{f \in \omega^\omega :$ there is an $f_\xi \in \mathcal{F}$ such that $f \leq^* f_\xi\}$. It is easy to see that Y is not Lindelöf with the $\mathcal{F}$-topology. So we only need to prove that, for any compact $K \subset \omega^\omega$, $K \cap Y$ is Lindelöf. Let $K \subset \omega^\omega$ be a compact subset. Let $f \in \omega^\omega$ be such that $K \subset K_f$. Note there are only countably many $f_\xi \in \mathcal{F}$ such that $f_\xi \leq^* f$; thus $K_f \cap Y$ is second countable in the $\mathcal{F}$-topology and so is $K \cap Y$. □

We can also obtain a Michael space with the $\mathcal{F}$-topology technique assuming $\mathfrak{b} = \mathfrak{d} = cov(\mathcal{M})$. The proof here is similar to the one in [8], which is a reworking of Alster's proof in [3]. We begin with a lemma that can be found in [8] (Lemma 2.10) and this is everything we need from $cov(\mathcal{M})$ here:

Lemma 3.5. *Let M be a compact metric space and $\mathcal{C}$ a closed covering of M with cardinality less than $cov(\mathcal{M})$. Then $\mathcal{C}$ has a countable subcovering.*

Proposition 3.6. *If $\mathfrak{b} = \mathfrak{d} = cov(\mathcal{M})$, then there is an $\mathcal{F} \subset \omega^\omega$ such that $X = Y = \omega^\omega$ and ρ equal to the $\mathcal{F}$-topology verify Theorem 2.4.*

Proof. Let $\mathcal{F} = \{f_\xi : \xi < \mathfrak{d}\}$ be a strong scale over ω^ω, *i.e.*, a well ordered family such that, for every $f \in \omega^\omega$, there is an $f_\xi \in \mathcal{F}$ such that $f \leq f_\xi$. It is easy to see that ω^ω is not Lindelöf with the $\mathcal{F}$-topology (simply take the $\{K_{f_\xi} : \xi < \mathfrak{d}\}$ covering). We have to show that every compact subset in the product topology is still compact in the $\mathcal{F}$-topology. Suppose not and let β be the first ordinal such that there is a compact K in the product topology such that $K \subset K_{f_\beta}$ and K is not Lindelöf in the $\mathcal{F}$-topology. Note that $K = \bigcup_{\xi \leq \beta} K_\xi \cap K$ and each $K_\xi \cap K$ for $\xi < \beta$ is compact in the product topology and, therefore, Lindelöf in the $\mathcal{F}$-topology by the minimality of β. Let $\mathcal{U}$ be an $\mathcal{F}$-open covering for $K \cap K_\beta$. For each $x \in K_\beta \cap K \setminus \bigcup_{\xi<\beta} K_\xi$, there is U_x an open set in the product topology such that $x \in U_x$ and $U_x \subset U$ for some $U \in \mathcal{U}$. Since ω^ω is hereditarily Lindelöf in the product topology, there is $\{x_n : n \in \omega\}$ such that $\bigcup_{n\in\omega} U_x \supset K_\beta \cap K \setminus \bigcup_{\xi<\beta} K_\xi$. Note that $K_\beta \cap K \setminus \bigcup_{n\in\omega} U_{x_n}$ is compact in the product topology. Thus, since $\beta < cov(\mathcal{M})$ and Lemma 3.5, $K_\beta \cap K \setminus \bigcup_{n\in\omega} U_{x_n}$ is the union of countably many Lindelöf spaces in the $\mathcal{F}$-topology.

□

In the previous result, one could try to assume $cov(\mathcal{M}) = \mathfrak{d}$, as in Moore's example ([14]). The problem appears in the proving that every original compact subset is $\mathcal{F}$-Lindelöf, since we no longer have an ordered family $\mathcal{F}$. We do not know if the result is true with such hypothesis.

4. Some applications

One easy way of applying Corollary 2.5 is when the compact subsets are countable:

Proposition 4.1. *If X is an uncountable first countable Lindelöf space such that every compact subspace is countable then X is not productively Lindelöf.*

PROOF. Because of Corollary 2.5, we only have to prove that X is Lindelöf at infinity. But this is true since every countable compact subset in a first countable space has countable character. □

Corollary 4.2. *If X is an uncountable first countable Lindelöf space of cardinality less than the continuum, then X is not productively Lindelöf.*

Thus, productive Lindelöfness can be easily destroyed by forcing:

Corollary 4.3. *If X is productively Lindelöf first countable, then after adding $> |X|$ many reals, X is no longer productively Lindelöf.*

It is easy to see that $\mathfrak{b}$ can be defined as

$$\min\{\kappa : \text{there is an uncountable } A \subset \omega^\omega \text{ such that } |A| = \kappa \text{ and,}$$

$$\text{for every compact } K \subset \omega^\omega,\ |K \cap A| < |A|\}$$

The next result just says that we could define such a cardinal $\mathfrak{b}(X)$ for every non-Menger space X and its value is bounded by $\mathfrak{b}$ (see [11] for the definitions of Menger and Hurewicz spaces):

Proposition 4.4. *Let X be a non-Menger Lindelöf space. Then there is a subspace $A \subset X$ such that $|A| < \mathfrak{b}$ and for every compact $K \subset X$, $|K \cap A| < |A|$.*

PROOF. Let $(f_\xi)_{\xi<\mathfrak{b}}$ be an unbounded sequence such that $f_\alpha \leq^* f_\beta$ if $\alpha \leq \beta$. Let $(\mathcal{A}_k)_{k\in\omega}$ be a sequence of open coverings such that, for each $k \in \omega$, $\mathcal{A}_k = (A^k_n)_{n\in\omega}$ is an increasing open covering for X and, for any $f : \omega \longrightarrow \omega$, $\bigcup_{k\in\omega} A^k_{f(k)}$ is not a covering for X. For each ξ, let

$$x_\xi \in X \setminus (\bigcup_{k\in\omega} A^k_{f_\xi(k)} \setminus \{x_\alpha < \xi\})$$

Define $A = \{x_\xi : \xi \in \mathfrak{b}\}$. Let K be a compact subset of X. Since K is compact, there is an $f \in \omega^\omega$ such that $K \subset A^k_{f(k)}$ for every $k \in \omega$. Since $(f_\xi)_{\xi<\omega_1}$ is unbounded, there is a $\xi < \omega_1$ such that $f_\xi \not\leq^* f$. Thus, $J_\theta = \{t \in \omega : f_\theta(t) > f(t)\}$ is infinite if $\theta > \xi$. Note that, if $t \in J_\theta$, $A^t_{f_\theta(t)} \supset A^t_{f(t)} \supset K$ and $x_\theta \notin A^t_{f_\theta(t)}$. Therefore, $|K \cap M| < \mathfrak{b}$. □

Even though the cardinal $\mathfrak{b}(X)$ can be defined for every non-Menger space X, its value may vary. Clearly, for the irrationals its value is $\mathfrak{b}$, but if one considers the irrationals with the subspace topology inherited from the Sorgenfrey line, such a space is non-Menger, and every compact subspace of it is countable. Therefore, the value of the cardinal would be ω_1 independently of the value of $\mathfrak{b}$.

Corollary 4.5. *Suppose $\mathfrak{b} = \omega_1$. Let X be a regular Lindelöf space such that X is Lindelöf at infinity. If X is productively Lindelöf, then X is Menger.*

Proof. Suppose X is not Menger. By the previous result, there is an uncountable $A \subset X$ such that $K \cap A$ is countable for every compact subset of X. Therefore, by Corollary 2.5, X is not productively Lindelöf. ☐

Corollary 4.6. *Productively Lindelöf regular spaces which are Lindelöf at infinity and the union of $\aleph_1$ compact sets are Menger.*

Proof. The previous corollary proved it, assuming $\mathfrak{b} = \aleph_1$. In [16], it is proved that Lindelöf spaces of size $< \mathfrak{b}$ are Hurewicz and hence Menger. ☐

This answers question 17 of [16].

In the following we generalize a concept considered by Alster in [2].

Definition 4.7. We say that X is an *Alster space* if, for every covering $\mathcal{G}$ of X by G_δ sets such that for every compact $K \subset X$, there is a finite $\mathcal{G}' \subset \mathcal{G}$ such that $K \subset \bigcup \mathcal{G}'$, there is a countable subfamily of $\mathcal{G}$ that is a covering for X.

Proposition 4.8 ([2]). *Alster spaces are productively Lindelöf. Under CH, productively Lindelöf spaces of weight $\leq \aleph_1$ are Alster spaces.*

Definition 4.9. Let X be a topological space. We call the *Alster degree* of X the least infinite cardinal κ such that, for every covering $\mathcal{G}$ of X by G_δ sets such that every compact subset of X is covered by finitely many elements of $\mathcal{G}$, $\mathcal{G}$ has a subcovering of cardinality less or equal to κ.

Proposition 4.10. *If X is Lindelöf at infinity and has Alster degree equal to $\aleph_1$, then X is not productively Lindelöf.*

Proof. Let $\mathcal{G} = \{G_\xi : \xi < \omega_1\}$ be a covering that witnesses the Alster degree of X. Moving to a subsequence if needed, let $x_\xi \in G_\xi \setminus \bigcup_{\eta<\xi} G_\eta$. Note that, for each compact $K \subset X$, $K \cap \{x_\xi : \xi < \omega_1\}$ is countable. Therefore, we can apply Corollary 2.5. ☐

Definition 4.11. Let X be a topological space. We denote by $K(X)$ the least cardinal κ such that there are $(K_\xi)_{\xi<\kappa}$ such that $X = \bigcup_{\xi<\kappa} K_\xi$ and each K_ξ is a compact subset of X.

Proposition 4.12. *Let X be a topological space. Then the Alster degree of X is less than or equal to $K(X)$.*

PROOF. Let $\mathcal{G}$ be a G_δ covering for X, closed under finite unions and such that each compact set is contained in one element of $\mathcal{G}$. Let $(K_\xi)_{\xi<\kappa}$ witnes that $K(X) = \kappa$. For each K_ξ, there is a $G_\xi \in \mathcal{G}$ such that $K_\xi \subset G_\xi$. Note that $(G_\xi)_{\xi<\kappa}$ is a covering for X. □

Proposition 4.13. *Let X be a Lindelöf at infinity space. Then the Alster degree of X is equal to $K(X)$.*

PROOF. Because of Proposition 4.12, we only have to prove that $K(X)$ is less or equal to the Alster degree of X. Let $\mathcal{K}$ be a covering of X by compact sets. By 2.2, for each $K \in \mathcal{K}$, there is a G_δ compact K' such that $K' \supset K$. Therefore, there is a subcovering $\mathcal{K}' \subset \mathcal{K}$ with cardinality equal to the Alster degree of X. □

In particular, we have the following characterization:

Corollary 4.14. *If X is Lindelöf at infinity, then X is Alster if, and only if, X is σ-compact.*

This was also proved in [16].

Thus, if there were a non σ-compact, Lindelöf at infinity, and productively Lindelöf space, it would give a negative answer to Alster's question.

We can now improve Corollary 4.6:

Corollary 4.15. *Productively Lindelöf spaces which are Lindelöf at infinity and are the union of $\aleph_1$ compact sets are σ-compact.*

PROOF. Spaces of size $\aleph_1$ have Alster degree $\leq \aleph_1$. By Corollary 4.10, a productively Lindelöf space which is Lindelöf at infinity cannot have Alster degree $= \aleph_1$, so must be Alster. By Corollary 4.14 it must be σ-compact. □

Arhangel′skiĭ's p-spaces ([4]) are Lindelöf at infinity.

Theorem 4.16. *CH implies productively Lindelöf regular p-spaces are σ-compact.*

PROOF. Recall that a paracompact p-space is a perfect pre-image of a metrizable space. In [16] it was noted that productive Lindelöfness is a perfect invariant, and so our productively Lindelöf p-space is a perfect pre-image of a productively Lindelöf metrizable space. σ-compactness is also a perfect invariant so it suffices to show that productively Lindelöf metrizable spaces are σ-compact. But that was done, assuming CH, implicitly in [13] and explicitly in [1]. □

Theorem 4.17. *There is a Michael space if and only if every productively Lindelöf Čech-complete regular space is σ-compact.*

Proof. Exactly like the proof of Theorem 4.16, since Čech-completeness is a perfect invariant of completely regular spaces, and in [5] we proved that there is a Michael space if and only if productively Lindelöf completely metrizable spaces are σ-compact. □

It was proved in [16] that *MA implies productively Lindelöf metrizable spaces are Hurewicz.* The usual argument yields that:

Theorem 4.18. *MA implies productively Lindelöf p-spaces are Hurewicz.*

However:

Theorem 4.19. *MA implies there is a Hurewicz set of reals which is not productively Lindelöf.*

Proof. If CH holds, productively Lindelöf sets of reals are σ-compact [13]. In ZFC, there are Hurewicz sets of reals that are not σ-compact [11]. If CH fails, since MA implies $\mathfrak{b} = 2^{\aleph_0}$, and sets of reals of size $< \mathfrak{b}$ are Hurewicz (folklore, see [16]), we can take any set of reals B of size $\aleph_1$. By Proposition 4.1, because compact subsets of B are countable, B will not be productively Lindelöf. □

We can strengthen Theorem 4.19:

Theorem 4.20. *MA imples there is a set of reals of size $\aleph_1$ which is not productively Lindelöf, yet all its finite powers are Hurewicz.*

Proof. In [7], the authors construct a non-σ-compact set of reals, such that every finite power of which is Hurewicz. Under CH, the subspace topology of that set is then not productively Lindelöf. If CH fails, since sets of reals of size $< 2^{\aleph_0}$ are 0-dimensional, any finite power of a set of size $\aleph_1$ is itself embedded in the Cantor set and hence the reals, and so will be Hurewicz. □

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