

# Lindelöf spaces which are indestructible, productive, or $D$

Leandro F. Aurichi<sup>1</sup> and Franklin D. Tall<sup>2</sup>

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## Abstract

We discuss relationships in Lindelöf spaces among the properties “indestructible”, “productive”, “ $D$ ”, and related properties.

## 1 Introduction

The question of what additional assumptions ensure that the product of two Lindelöf spaces is Lindelöf is natural and well-studied. See e.g., [24], [25], [2], [3], [30], [31]. The question of which topological properties are preserved by which kinds of forcing is also a natural one. See e.g., [12], [39], [17], [19], [38], etc. The question of whether a Lindelöf space remains Lindelöf after countably closed forcing is particularly interesting because of its connection with the classic problem of whether Lindelöf spaces with points  $G_\delta$  consistently have cardinality  $\leq 2^{\aleph_0}$  [39]. We need some definitions.

**Definition.** A space is *indestructibly Lindelöf* if it is Lindelöf in every countably closed forcing extension.

Note that indestructibly Lindelöf spaces are Lindelöf.

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$D$ -spaces were introduced in [11].

**Definition.** A space  $X$  is  $D$  if for every *neighbourhood assignment*  $\{V_x\}_{x \in X}$ , i.e. each  $V_x$  is an open set containing  $x$ , there is a closed discrete  $Y \subseteq X$  such that  $\{V_x\}_{x \in Y}$  covers  $X$ .  $Y$  is called a *kernel* of the neighbourhood assignment.

The question raised in [11] of whether every Lindelöf space is a  $D$ -space has been surveyed in [13] and [16]. It has recently been the subject of much research. In [4], the first author established many connections between the  $D$  property, topological games, and selection properties. In this paper, we examine Lindelöf indestructibility and connections of it, selection principles, and the  $D$  property with preservation of Lindelöfness under products.

**Definition** [7]. A space  $X$  is *productively Lindelöf* if  $X \times Y$  is Lindelöf for any Lindelöf space  $Y$ .

**Definition.** A space is *indestructibly productively Lindelöf* if it is productively Lindelöf in any extension by countably closed forcing.

**Definition** [3], [7]. A space is *Alster* if every cover by  $G_\delta$  sets that covers each compact set finitely includes a countable subcover.

**Definition.** A space is *indestructibly  $D$*  if it remains  $D$  after countably closed forcing.

We shall deal with three selection principles in this paper. These principles have a variety of equivalent definitions — see e.g. [35]. Here are the first two. The third — *Hurewicz* — is defined in Section 4.

**Definition.** A space is *Menger* if for each sequence  $\{\mathcal{U}_n\}_{n < \omega}$  of open covers, such that each finite union of elements of  $\mathcal{U}_n$  is a member of  $\mathcal{U}_n$ , there are  $U_n \in \mathcal{U}_n$ ,  $n < \omega$ , such that  $\{U_n : n < \omega\}$  is an open cover. A space is *Rothberger* if for each sequence  $\{\mathcal{U}_n\}_{n < \omega}$  of open covers, there are  $U_n \in \mathcal{U}_n$ , such that  $\{U_n : n < \omega\}$  is an open cover.

Clearly every Rothberger space is Menger. Previously known results include:

**Lemma 1** [35]. *Every Rothberger space is indestructibly Lindelöf.*

**Lemma 2** [3]. *Every Alster space is productively Lindelöf;  $CH$  implies every productively Lindelöf  $T_3$  space of weight  $\leq \aleph_1$  is Alster. Alster metrizable spaces are  $\sigma$ -compact.*

## 2 $D$ -spaces

We shall now see what we can say about  $D$ -spaces. Theorem 4 and Corollary 5 are the important results in this section. No result resembling Corollary 6 was previously known.

**Lemma 3** [4]. *Every Menger space is  $D$ .*

**Theorem 4.** *Every Alster space is Menger.*

*Proof.* Let  $\{\mathcal{U}_n\}_{n<\omega}$  be a sequence of open covers of  $X$ , each closed under finite unions. Let  $\mathcal{G}$  be the set of all  $\bigcap_{n<\omega} U_n$ 's, where  $U_n \in \mathcal{U}_n$ . Let  $K$  be any compact subspace of  $X$ . Then for each  $n < \omega$ ,  $K$  is included in some  $U_n \in \mathcal{U}_n$ . Thus  $K$  is included in some  $G \in \mathcal{G}$ . Since  $X$  is Alster, there are  $\{H_k\}_{k<\omega}$  in  $\mathcal{G}$  such that  $\bigcup_{k<\omega} H_k$  covers  $X$ . Let  $H_k = \bigcap_{n<\omega} U_{nk}$ , where  $U_{nk} \in \mathcal{U}_n$ . Then  $\{U_{nn}\}_{n<\omega}$  covers  $X$ , since  $H_n \subseteq U_{nn}$ . Thus, since each  $U_{nn} \in \mathcal{U}_n$ ,  $X$  is Menger.  $\square$

**Corollary 5.** *Every Alster space is  $D$ .*

**Corollary 6.**  *$CH$  implies every productively Lindelöf  $T_3$  space which is either first countable or separable is  $D$ .*

*Proof.* First countable Lindelöf Hausdorff spaces have cardinality and hence weight  $\leq 2^{\aleph_0}$ ; separable regular spaces have weight  $\leq 2^{\aleph_0}$ .  $\square$

## 3 Indestructibly productively Lindelöf spaces

“Indestructibly productively Lindelöf” is much harder to understand than is “indestructibly Lindelöf”. In a previous version of this note, we prematurely claimed that indestructibly productively Lindelöf  $T_3$  spaces are Alster. This may well be true, but, at the moment, we do not have a proof. We can, however, prove this for metrizable spaces, which is the key result of this section.

**Theorem 7.** *A metrizable space is indestructibly productively Lindelöf if and only if it is  $\sigma$ -compact.*

*Proof.* The backward direction is routine, since “ $\sigma$ -countably-compact” and “metrizable” are both preserved by countably closed forcing, and so hence is “ $\sigma$ -compact metrizable”. It is well-known that  $\sigma$ -compact spaces are productively Lindelöf. For the other direction, first recall:

**Lemma 8** ([14, 4.4J]). *Every separable metrizable space is a perfect image of a 0-dimensional separable metrizable space.*

We claim that if we prove Theorem 7 for 0-dimensional spaces, it will follow for all spaces. For suppose  $X$  is an indestructibly productively Lindelöf metrizable space and  $X'$  is its 0-dimensional perfect pre-image by a map  $f$ . Note that a space with a countable base has no new open or closed sets in a countably closed extension. Thus  $f$  remains continuous and closed in the extension. It may not be perfect, but inverse images of points are Lindelöf. Let  $Y$  be Lindelöf in the extension. Then  $f \times \text{id}_Y$  also is continuous, closed, and has inverse images of points Lindelöf. Since  $X \times Y$  is Lindelöf, it follows that  $X' \times Y$  is Lindelöf. Thus  $X'$  is indestructibly productively Lindelöf. It is then  $\sigma$ -compact and therefore so is  $X$ . Thus without loss of generality, we shall assume our space  $X$  is 0-dimensional, and hence can be considered as a subspace of the Cantor set  $\mathbb{K}$ .

Collapse  $2^{\aleph_0}$  to  $\aleph_1$  by countably closed forcing. Then  $X$  remains productively Lindelöf. By Lemma 2, it is  $\sigma$ -compact in the extension. As noted above,  $\mathbb{K}$  has no new closed sets and hence no new  $F_\sigma$ 's. Thus if  $X = \bigcup_{n < \omega} F_n$  in the extension, where the  $F_n$ 's are compact and hence closed subspaces of  $\mathbb{K}$ , then the  $F_n$ 's are actually in the ground model and hence compact there as well.  $\square$

There are some other conditions that imply indestructibly productively Lindelöf:

**Theorem 9.** *Every Lindelöf space which either is scattered or is a  $P$ -space or is indestructibly Lindelöf and  $\sigma$ -compact is indestructibly productively Lindelöf.*

*Proof.* This follows easily since:

**Lemma 10** [20]. *The Lindelöfness (and scatteredness) of a scattered space is preserved by any forcing.*

**Lemma 11** [7]. *Every Lindelöf space which either is scattered or is a  $P$ -space is Alster.*

**Lemma 12** [35]. *Lindelöf  $P$ -spaces are Rothberger and hence indestructible.*

Clearly the  $P$ -property ( $G_\delta$ 's open) is preserved by countably closed forcing. That indestructibly Lindelöf  $\sigma$ -compact spaces are indestructibly productively Lindelöf follows from the proof of Theorem 7.  $\square$

We don't know whether indestructibly Lindelöf or productively Lindelöf implies  $D$ . Indestructibly productively Lindelöf does [41]. Note that an indestructibly Lindelöf non- $D$  space remains non- $D$  in any countably closed forcing extension.

**Theorem 13.** *Suppose  $X$  is Lindelöf,  $D$ , and countably tight. Then  $X$  is indestructibly Lindelöf if it is indestructibly  $D$ .*

*Proof.* In [39] it is shown that after countably closed forcing, Lindelöf countably tight spaces retain countable extent. But  $D$ -spaces with countable extent are Lindelöf.  $\square$

**Theorem 14.** *Suppose  $X$  is Lindelöf,  $D$ ,  $|X| \leq \aleph_1$ . Then  $X$  is both indestructibly Lindelöf and indestructibly  $D$ .*

*Proof.* That Lindelöf spaces of size  $\leq \aleph_1$  are indestructible was proved in [39]. Suppose  $\{\dot{V}_{x_\alpha}\}_{\alpha < \omega_1}$  is a neighbourhood assignment in the extension; without loss of generality, we may assume each  $\dot{V}_{x_\alpha}$  is a ground model open set. Given an arbitrary condition  $p$  forcing all this, take below  $p$  a descending sequence of conditions  $\{p_\alpha\}_{\alpha < \omega_1}$  deciding  $\dot{V}_{x_\alpha}$ . The resulting  $V_{x_\alpha}$ 's form a neighbourhood assignment in the ground model. It had a countable kernel  $\{x_{\alpha_n}\}_{n < \omega}$ . Let  $\alpha_\omega \geq$  each  $\alpha_n$ . Then  $p_{\alpha_\omega}$  forces  $(\{x_{\alpha_n}\}_{n < \omega})^\smallfrown$  is a kernel for  $\{\dot{V}_{x_\alpha}\}_{\alpha < \omega_1}$ .  $\square$

**Corollary 15.**  *$CH$  implies productively Lindelöf first countable  $T_3$  spaces are indestructibly  $D$ .*

Productively Lindelöf  $D$ -spaces are not necessarily indestructibly  $D$ : consider the usual product topology on  $2^{\omega_1}$ . Adding a Cohen subset of  $\omega_1$  with countable conditions makes  $2^{\omega_1}$  non-Lindelöf [39], but countably closed forcing preserves countable compactness. Since countably compact plus  $D =$  compact, we see that the forcing does not preserve  $D$ . Thus countably closed forcing does not preserve Menger or Alster. Neither does it preserve productively Lindelöf. To see this, again consider adding a Cohen subset of  $\omega_1$  with countable conditions.  $2^{\omega_1}$  is productively Lindelöf in the ground model; its weight is  $\aleph_1$ , so in the extension, if it were productively Lindelöf, it would be Alster, since  $CH$  holds.

Since Rothberger implies both indestructibly Lindelöf and Menger, one might wonder if it is strong enough to imply productively Lindelöf. It is not; see Section 9 below. Thus Lindelöf productivity is not a necessary condition for  $D$ -ness in Lindelöf spaces. Alster does not imply Rothberger, since Alster is equivalent to  $\sigma$ -compact in metrizable spaces [3], but Rothberger subsets of the real line have strong measure zero — see e.g. [26], where they are called  $C'''$  sets. Similarly, indestructibly productively Lindelöf does not imply Rothberger — consider the closed unit interval.

Indestructibly Lindelöf spaces need not be productively Lindelöf; a Bernstein (totally imperfect) set of reals provides a counterexample [25]. Alster does not imply indestructibly productively Lindelöf, since  $\sigma$ -compact spaces are Alster [3].

Among the properties we have considered so far, the interesting open questions (say for  $T_3$  spaces) are:

1. *Do any of Lindelöf, indestructibly Lindelöf, productively Lindelöf imply  $D$ ?*

2. Does productively Lindelöf imply Alster [7]? (This question was first asked in [3], with different terminology.) Does indestructibly productively Lindelöf imply Alster?
3. Are indestructibly Lindelöf, productively Lindelöf spaces indestructibly productively Lindelöf?
4. Are indestructibly Lindelöf  $D$ -spaces indestructibly  $D$ ?

## 4 Productively Lindelöf completely metrizable spaces

The question of whether productively Lindelöf spaces are Alster reduces in the metrizable case to whether they are  $\sigma$ -compact. The second author examines this in detail in [37]; here we shall mainly confine ourselves to the completely metrizable case. Recall the famous problem of E. Michael which asks whether there is a Lindelöf space whose product with the space  $\mathbb{P}$  of irrationals is not Lindelöf. See [24], [25], [27]. We shall prove:

**Theorem 16.** *The following assertions are equivalent:*

- a) Every completely metrizable productively Lindelöf space is Menger,
- b) Every completely metrizable productively Lindelöf space is Alster,
- c) Every completely metrizable productively Lindelöf space is  $\sigma$ -compact,
- d) There is a Lindelöf space  $X$  such that  $X \times \mathbb{P}$  is not Lindelöf.

*Proof.* As mentioned earlier, a metrizable space is Alster if and only if it is  $\sigma$ -compact, if and only if it is indestructibly productively Lindelöf. To show a), b), c) equivalent, then, it suffices to prove a) implies c). We note that every productively Lindelöf space is Lindelöf, and hence, if metrizable, is separable.

We next need:

**Lemma 17** [23, I.7.8]. *Every 0-dimensional separable metrizable Čech-complete space is homeomorphic to a closed subspace of  $\mathbb{P}$ , the space of irrationals, considered as  $\omega^\omega$ .*

These yield:

**Lemma 18.** *If there is a productively Lindelöf completely metrizable space which is not  $\sigma$ -compact, then there is one included in  $\mathbb{P}$ .*

*Proof.* As in the proof of Theorem 7, if  $X'$  maps perfectly onto a productively Lindelöf  $X$ , then  $X'$  is productively Lindelöf. Next, recall that a perfect image of a  $\sigma$ -compact space is  $\sigma$ -compact. If then  $X$  is not  $\sigma$ -compact, then neither is  $X'$ . Finally, the perfect pre-image of a completely metrizable space is completely metrizable.  $\square$

Hurewicz [18] proved that analytic (and, in particular,  $G_\delta$ ) sets of reals are Menger if and only if they are  $\sigma$ -compact. Thus, we see that, assuming a), 0-dimensional completely metrizable productively Lindelöf spaces are  $\sigma$ -compact. The non-0-dimensional case then follows.

Having established that a) implies c) we next prove that c) is equivalent to d). Since  $\mathbb{P}$  is not  $\sigma$ -compact, it follows that if every completely metrizable productively Lindelöf space is  $\sigma$ -compact, then there must be a Lindelöf space  $X$  with  $X \times \mathbb{P}$  not Lindelöf. Conversely, assume there is a completely metrizable productively Lindelöf space  $Y$  which is not  $\sigma$ -compact. Let  $X$  be any Lindelöf space. Claim:  $X \times \mathbb{P}$  is Lindelöf. Recall Hurewicz's Theorem:

**Lemma 19**([23, 7.10]). *If  $Y$  is a completely metrizable Lindelöf space which is not  $\sigma$ -compact, then  $Y$  includes a closed copy of  $\mathbb{P}$ .*

It follows that  $X \times \mathbb{P}$  is a closed subspace of  $X \times Y$ . Since  $X \times Y$  is Lindelöf, so is  $X \times \mathbb{P}$ .  $\square$

Cardinal invariants of the continuum are closely related to Michael's problem.

**Definition.** Partially order  ${}^\omega\omega$  by  $f \leq^* g$  if  $f(n) \leq g(n)$  for all but finitely many  $n$ .  $\mathfrak{b}$  is the least cardinal such that for every  $\mathcal{F} \subseteq {}^\omega\omega$  of size  $< \mathfrak{b}$ , there is a  $g \in {}^\omega\omega$  such that for each  $f \in \mathcal{F}$ ,  $f \leq^* g$ .  $\mathfrak{d}$  is the least cardinal  $\delta$  such that there is a family  $\mathcal{F}$  of size  $\delta$  included in  ${}^\omega\omega$  such that for every  $f \in {}^\omega\omega$ , there is a  $g \in \mathcal{F}$ , such that  $f \leq^* g$ .  $\text{cov}(\mathcal{M})$  is the least cardinal  $\delta$  such that  ${}^\omega\omega$ , identified with  $\mathbb{P}$ , is the union of  $\delta$  nowhere dense sets. A  $\lambda$ -**scale** is a subset  $S$  of  ${}^\omega\omega$  of size  $\lambda$  such that  $<^*$  (i.e.,  $\leq^*$ , but not for all but finitely many  $n$  equal) well-orders  $S$  and each  $f \in {}^\omega\omega$  is less than some member of  $S$ .

**Lemma 20** [10].  $\mathfrak{b} = \aleph_1$  implies there is a Lindelöf regular space  $X$  such that  $X \times \mathbb{P}$  is not Lindelöf.

**Lemma 21** [27].  $\mathfrak{d} = \text{cov}(\mathcal{M})$  implies there is a Lindelöf regular space  $X$  such that  $X \times \mathbb{P}$  is not Lindelöf.

**Corollary 22.**  $\mathfrak{b} = \aleph_1$  or  $\mathfrak{d} = \text{cov}(\mathcal{M})$  implies every productively Lindelöf, completely metrizable space is  $\sigma$ -compact.

It is interesting to wonder whether there is a test space for whether productively Lindelöf metrizable spaces are  $\sigma$ -compact, as is provided by  $\mathbb{P}$  in the completely metrizable case. Under  $CH$ , by Lemma 2 every productively Lindelöf metrizable space is  $\sigma$ -compact. This is proved explicitly in [2]. If there were a productively Lindelöf, metrizable, non- $\sigma$ -compact space, it would be an example of a productively Lindelöf non-Alster space, and of a productively Lindelöf, indestructibly Lindelöf space which would not be indestructibly productively Lindelöf.

In a previous version of this note, we claimed that  $\mathfrak{d} = \aleph_1$  implies productively Lindelöf metrizable spaces are  $\sigma$ -compact, improving the  $CH$  result referred to above. The referee pointed out an error in our proof, which we have been unable to fix. We are no longer confident of the truth of our claim.

**Problem.** Does  $\mathfrak{d} = \aleph_1$  imply productively Lindelöf metrizable spaces are  $\sigma$ -compact?

As a consolation prize, we shall prove a weaker assertion.

**Definition.** A  $\gamma$ -cover of a space is a countably infinite open cover such that each point is in all but finitely many members of the cover. A space is **Hurewicz** if, given a sequence  $\{\mathcal{U}_n : n \in \omega\}$  of  $\gamma$ -covers, there is for each  $n$  a finite  $\mathcal{V}_n \subseteq \mathcal{U}_n$  such that either  $\{\bigcup \mathcal{V}_n : n \in \omega\}$  is a  $\gamma$ -cover, or else for some  $n$ ,  $\mathcal{V}_n$  is a cover.

**Theorem 23.**  $\mathfrak{d} = \aleph_1$  implies every productively Lindelöf metrizable space is Hurewicz.

The Hurewicz property fits strictly between Menger and  $\sigma$ -compact. See e.g. [22], [44], and [42]. In a successor [37] to this paper, the second author proves that Alster implies Hurewicz. There are a number of equivalent definitions of Hurewicz – see [22], [44], [6], [37].

It will be convenient to work with one of them. To avoid relying on the unpublished [37], we shall temporarily call this property *Hurewicz\**:

**Definition.** A Lindelöf  $T_3$  space is **Hurewicz\*** if and only if every Čech-complete  $Y \supseteq X$  includes a  $\sigma$ -compact  $Z \supseteq X$ .

Banach and Zdomskyy [6] prove that:

**Lemma 24.** *Hurewicz\** is equivalent to Hurewicz in separable metrizable spaces.

We generalized this to Lindelöf  $T_3$  spaces in [37], but their version is all we need here. We next observe:

**Lemma 25.** A  $T_{3\frac{1}{2}}$  perfect image of a *Hurewicz\**  $T_{3\frac{1}{2}}$  space is *Hurewicz\**.



*Proof.* Let  $p : X$  onto  $X_0$  be perfect. Let  $Y_0$  be a Čech-complete space including  $X_0$ . Then the closure  $\overline{X_0}$  of  $X_0$  in  $Y_0$  is also Čech-complete. Then  $\beta\overline{X_0}$  is a compactification of  $X_0$ . Recall:

**Lemma 26** ([14, 3.6.6]). *For every compactification  $\alpha T$  of a  $T_{3\frac{1}{2}}$  space  $T$  and every continuous map  $f : S \rightarrow T$  of a  $T_{3\frac{1}{2}}$  space  $S$  to the space  $T$ , there is a continuous extension  $F : \beta S \rightarrow \alpha T$  over  $\beta S$  and  $\alpha T$ .*

Thus we may extend  $p : X \rightarrow X_0$  to  $P : \beta X \rightarrow \beta\overline{X_0}$ . Let  $Y = P^{-1}(\overline{X_0})$ . Then  $Y$  is a Čech-complete space including  $X$ , since Čech-completeness is a perfect invariant for  $T_{3\frac{1}{2}}$  spaces [14]. Let  $W$  be  $\sigma$ -compact,  $X \subseteq W \subseteq Y$ . Then  $P(W)$  is  $\sigma$ -compact,  $X_0 \subseteq P(W) \subseteq Y_0$ .  $\square$

By Lemmas 8, 24, and 25, and since it is easy to see that productive Lindelöfness is a perfect invariant, we may conclude that if there is a productively Lindelöf space which is not Hurewicz, there is one included in  $\mathbb{P}$ . Furthermore, by the following result of Reclaw [32], we may further assume that there is a productively Lindelöf  $X \subseteq \mathbb{P}$  such that  $X$  is not included in any  $\sigma$ -compact subspace of  $\mathbb{P}$ .

**Lemma 27** [32]. *A 0-dimensional subset  $X$  of  $\mathbb{P}$  is Hurewicz if and only if every homeomorph of  $X$  included in  $\mathbb{P}$  is included in a  $\sigma$ -compact subspace of  $\mathbb{P}$ .*

Let  $\{f_\alpha : \alpha < \omega_1\}$  be a dominating family for  ${}^\omega\omega$ , thinned out to form a scale. For each  $\alpha < \omega_1$ , let  $x_\alpha \in X$  be such that  $x_\alpha \not\leq^* f_\beta$ , for every  $\beta < \alpha$ . There always is such an  $x_\alpha$ , else  $X$  would be included in a  $\sigma$ -compact subspace of  $\mathbb{P}$ . Considering  $\mathbb{P}$  as a subspace of  $[0, 1]$ , let  $Y' = [0, 1] - X$ . Let  $Y = Y' \cup \{x_\alpha : \alpha < \omega_1\}$ . Strengthen the topology on  $Y$  by making all the  $x_\alpha$ 's isolated. Then claim  $Y$  is still Lindelöf. For if  $V \supseteq Y'$  is open in  $[0, 1]$ , then  $[0, 1] - V$  is compact in  $[0, 1]$  and included in  $X$ . Then some  $f_\alpha$  bounds it. Since the  $f_\alpha$ 's form a scale, none of the  $x_\beta$ 's, for  $\beta \geq \alpha$  are  $\leq^* f_\alpha$ . Therefore there are only countably many  $x_\alpha$ 's in  $[0, 1] - V$ . Then any open cover of  $Y$  will include countably many open sets which cover all but countably many members of  $Y$ . The usual argument shows that  $X \times Y$  is not Lindelöf, since  $\{\langle x_\alpha, x_\alpha \rangle : \alpha < \omega_1\}$  is uncountable closed discrete.  $\square$

## 5 Other productive properties

There are some other properties we may productively consider.

**Definition.** A space is *powerfully Lindelöf* if its  $\omega$ th power is Lindelöf. A space is *finitely powerfully Lindelöf* if all of its finite powers are Lindelöf. (Finitely powerfully Lindelöf spaces are called  $\varepsilon$ -spaces in [15].)

**Lemma 28** [3]. *Alster spaces are powerfully Lindelöf.*

**Lemma 29** [3]. *Productively Lindelöf spaces are finitely powerfully Lindelöf.*

Przymusiński [30] has constructed a finitely powerfully Lindelöf space that is not powerfully Lindelöf. Michael [25] constructed a subset  $M$  of the real line and a Lindelöf space such that the product of the two was not Lindelöf. Thus  $M$  is powerfully Lindelöf but not productively Lindelöf. He also proved:

**Lemma 30** [25]. *If  $X^\omega$  is normal, then  $X \times \mathbb{P}$  is normal.*

On the other hand,

**Lemma 31** [34]. *Suppose  $X$  is Lindelöf regular and  $Y$  is separable metrizable. Then  $X \times Y$  is normal if and only if  $X \times Y$  is Lindelöf.*

It follows that:

**Theorem 32.** *If  $X$  is regular and powerfully Lindelöf, then  $X \times \mathbb{P}$  is Lindelöf.*

Michael also raised the following question (the earliest reference we have found is [31]), which is still unsolved:

5. *Are productively Lindelöf spaces powerfully Lindelöf?*

We can give a partial answer:

**Definition.** A space is *productively FC-Lindelöf* if its product with every first countable Lindelöf  $T_3$  space is Lindelöf.

**Theorem 33.** *CH implies if  $X$  is first countable  $T_3$ , and productively FC-Lindelöf, then  $X$  is Alster, and hence  $X$  is powerfully Lindelöf.*

*Proof.* Note that  $X$  is Lindelöf and hence has weight  $\leq 2^{\aleph_0}$ . Assuming *CH*, given a non-Alster Lindelöf  $T_3$  space  $X$  of weight  $\leq \aleph_1$ , Alster [3] constructs a space  $Y' = P \cup A$  such that:

1.  $Y'$  is Lindelöf,
2.  $X \times Y'$  is not Lindelöf,
3.  $P$  is a set of isolated points,
4.  $Y'$  is a subspace of a space  $Y$  in which each  $a \in A$  has a countable neighbourhood base.

But then  $Y'$  is first countable. □

An example of a finitely powerfully Lindelöf space whose product with  $\mathbb{P}$  is not Lindelöf can be constructed by Michael's original construction. Recall that construction produces from  $CH$  an uncountable subset  $C$  of  $\mathbb{R}$  concentrated on the rationals.  $C$ , as a subspace of the Michael line, is Lindelöf, yet  $C \times \mathbb{P}$  is not. We simply need such a  $C$  with  $C^n$  Lindelöf for every  $n$ . Michael in fact constructs such a  $C$  and hence such a space from  $CH$  in [25]. In an earlier version of this note, we claimed we could get this from  $\mathfrak{b} = \aleph_1$ , but B. Tsaban found an error in the proof. This claim has now been verified in [41].

## 6 The Rothberger property and concentrated sets

There are some more points concerning the Rothberger property worth noting.

**Definition.** We say that a topological space is *concentrated* on  $Y \subseteq X$  if, for every open set  $U$  such that  $U \supseteq Y$ ,  $X \setminus U$  is countable.

**Theorem 34** (folklore). *If  $X$  is concentrated on a Rothberger (Menger) subspace, then  $X$  is Rothberger (Menger).*

*Proof.* We will prove the Rothberger case; the Menger case is analogous. Note that  $X$  is Lindelöf. Let  $(\mathcal{U}_n)_{n \in \omega}$  be a sequence of open coverings for  $X$ . Let  $Y$  be a Rothberger subspace such that  $X$  is concentrated on it. Let  $(U_{2n})_{n \in \omega}$  be a covering for  $Y$  such that each  $U_{2n} \in \mathcal{U}_{2n}$ . Let  $\{x_n : n \in \omega\} = X \setminus \bigcup_{n \in \omega} U_{2n}$ . For each  $n \in \omega$ , pick  $U_{2n+1} \in \mathcal{U}_{2n+1}$  such that  $x_n \in U_{2n+1}$ . Note that  $(U_n)_{n \in \omega}$  is a covering for  $X$ .  $\square$

**Definition.** A space is *Lusin* if every nowhere dense set is countable.

**Corollary 35.** *Every separable Lusin space is Rothberger and, therefore,  $D$ .*

*Proof.* Observe that a separable Lusin space is concentrated on a countable set.  $\square$

Separability cannot be dispensed with. A *Sierpiński set* is an uncountable set of reals which has countable intersection with every null set. Sierpiński sets exist under  $CH$ ; they are Lusin in the density topology on the real line, indeed the null sets coincide with the first category sets – see [40] for details. Rothberger sets have (strong) measure 0 [26], so Sierpiński sets cannot be Rothberger in the usual topology on the real line and hence not in any strengthening of that.

Michael's space is concentrated on the rationals, so is Rothberger. Thus it is consistent that a Rothberger space (therefore a  $D$ -space) need not be

productively Lindelöf even if the products are taken only with well-behaved spaces such as  $\mathbb{P}$ .

## 7 Elementary submodels

A similar argument to the forcing one following Corollary 15 shows that elementary submodels do not preserve  $D$  or Lindelöf. Let  $M$  be a countably closed elementary submodel of some  $H_\theta$ , where  $\theta$  is a sufficiently large regular cardinal, with the compact space  $X$  in  $M$ . Then the space  $X_M$  defined in [21], namely the topology on  $X \cap M$  generated by  $\{U \cap M : U \in M \text{ and } U \text{ open in } X\}$  is countably compact [21]. On the other hand, if we take  $X$  to be e.g.  $2^{2^{\aleph_0}}$  and  $|M| = 2^{\aleph_0}$ ,  $X_M$  will not be compact [21], and hence not  $D$  nor Lindelöf.

However, we have:

**Theorem 36.** *If  $X$  is a first countable  $T_2$   $D$ -space, and  $M$  is a countably closed elementary submodel containing  $X$ , then  $X_M$  is  $D$ .*

*Proof.* In such a situation,  $X_M$  is a closed subspace of  $X$  [21]. Closed subspaces of  $D$ -spaces are easily seen to be  $D$ -spaces.  $\square$

**Theorem 37.** *If  $X$  is  $T_2$  and of pointwise countable type and hereditarily  $D$ , then  $X_M$  is  $D$ .*

*Proof.* By [21], if  $X$  is  $T_2$  and of pointwise countable type, then  $X_M$  is a perfect image of a subspace of  $X$ . By [8], perfect images of  $D$ -spaces are  $D$ .  $\square$

Alster [3] asked whether if  $CH$  holds and every closed subspace of  $X$  of weight  $\leq \aleph_1$  is Alster, then  $X$  must be Alster. We can prove this for first countable spaces:

**Theorem 38.** *Suppose  $X$  is first countable  $T_2$  and each closed subspace of  $X$  of weight  $\leq 2^{\aleph_0}$  is Alster. Then  $X$  is Alster.*

*Proof.* Since first countable  $T_2$  Lindelöf spaces have cardinality no more than that of the continuum, it suffices to show  $X$  is Lindelöf. Take a countably closed elementary submodel  $M$  of some sufficiently large  $H_\theta$  which contains  $X$  and its topology. Then  $X \cap M (= X_M)$  is a closed subspace of  $X$  [21]. Therefore  $X \cap M$  is Alster and hence Lindelöf. But then by [21],  $X$  is Lindelöf.  $\square$

Similar arguments clearly work if we replace “Alster” by “ $\sigma$ -compact” or other strengthenings of “Lindelöf” in the statement of the Theorem.

**Corollary 39.** *CH implies that if  $X$  is first countable  $T_3$  and each closed subspace of size  $\leq \aleph_1$  is productively FC-Lindelöf, then  $X$  is Alster.*

**Corollary 40.** *CH implies that if  $X$  is metrizable and each closed subspace of  $X$  of size  $\leq \aleph_1$  is productively FC-Lindelöf, then  $X$  is  $\sigma$ -compact.*

*Proof.* In a first countable space, every subspace of size  $\leq 2^{\aleph_0}$  has weight  $\leq 2^{\aleph_0}$ .  $\square$

## 8 Other forcings

We can also consider preservation of Lindelöf and  $D$  by other kinds of forcing. For example, it is known that a space is Lindelöf in a Cohen or random real extension if and only if it is in the ground model [12], [17], [35], and [38]. The situation for  $D$  is more complicated; in [5] it is shown that a Lindelöf space  $X$  becomes a  $D$ -space in an extension by more than  $|X|$  Cohen reals. It follows immediately from Lemma 3 and [35] that this can be improved to:

**Theorem 41.** *Adding  $\aleph_1$  Cohen reals makes a Lindelöf space  $D$ .*

*Proof.* By [35] that makes the space Rothberger and hence Menger; by Lemma 3 it is hence  $D$ .  $\square$

We do not know the answer to the following:

**Problem 6.** Suppose  $X$  is Lindelöf in the ground model and  $D$  in a random real extension. Must  $X$  be  $D$  in the ground model?

We conjecture “yes”, at least for 0-dimensional  $X$ . In [35], it is shown that if a space is Menger in a random extension, then it is Menger in the ground model.

## 9 Examples and implications

Figure 1 illustrates the relationships among the properties we have discussed. For convenience, we assume  $T_3$  throughout the diagram and examples. Zdomskyy [45] proved that if  $\mathfrak{u} < \mathfrak{g}$ , then Rothberger spaces are Hurewicz. An easier proof is in [44]. Tall [41] proved this from Borel’s Conjecture. Moore’s  $L$ -space [28] is Rothberger and Hurewicz [35], but is not productively Lindelöf. To see this, we note that Tsaban and Zdomskyy [43] have shown that there is an  $n \in \omega$  such that  $L^n$  is not Lindelöf, so  $L$  is not powerfully Lindelöf. Let  $n_0$  be the least such  $n$ . Then  $L^{n_0-1}$  is Lindelöf, but  $L \times L^{n_0-1}$  is not. We do not know whether, as claimed in [35],  $L^2$  is not Lindelöf. In an earlier version of this note, we asked whether indestructibly productively

Lindelof spaces are D. In fact, they are Hurewicz [41]. Also in [41] we show that indestructibly productively Lindelöf spaces are powerfully Lindelöf. A number next to a solid arrow means that the example with that number from the list below shows that the arrow does not reverse. A number next to a broken dashed arrow means that that example shows that the implication does not hold. A dotted arrow indicates that the implication holds under the indicated hypothesis.

The numbers refer to the following examples:

1. Moore's  $L$ -space [28].
2.  $[0, 1]$ .
3. The space  $\mathbb{P}$  of irrationals.
4.  $2^{\omega_1}$ .
5. A Hurewicz (and hence Menger) subspace of the real line which is not  $\sigma$ -compact (and hence not Alster) [22], [44], [42].
6. Michael's space [25].
7. The one-point Lindelöfication of the discrete space of size  $\aleph_1$ .
8. A Bernstein set [25].
9. Przymusiński's space [30].
10. Another example in [30].
11. The *Sorgenfrey line* is well-known to be Lindelöf, have closed sets  $G_\delta$ , and to have non-Lindelöf square; on the other hand, the product of a Lindelöf space with closed sets  $G_\delta$  with a separable Lindelöf space (such as  $\mathbb{P}$ ) is Lindelöf [31].
12. A Menger subspace of the real line which is not Hurewicz [9], [44], [42].
13. The subspace of the Michael line obtained from a set concentrated on the rationals.

Some of the most interesting problems from the diagram are:

- A. Is there a ZFC example of a Lindelöf space whose product with  $\mathbb{P}$  is not Lindelöf?
- B. Is there a productively Lindelöf space which is not powerfully Lindelöf?
- C. Is there a productively Lindelöf space which is not Alster?

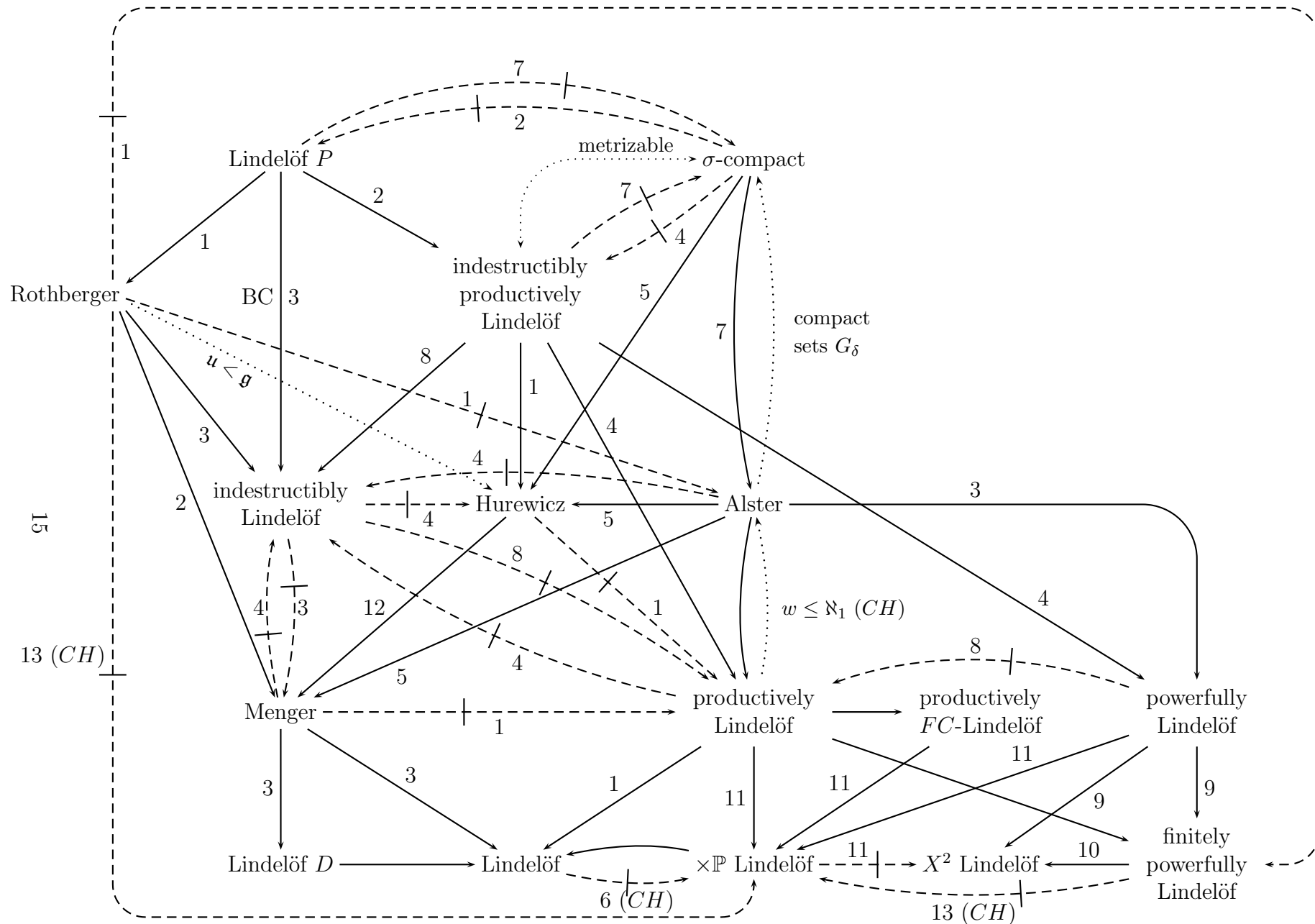


Figure 1: The relationships among various properties discussed.

**Remark.** Since this paper was submitted, there have been several developments worth noting:

1. There is an easy proof that CH implies productively Lindelöf spaces are Menger, and hence  $D$  [36].
2. The completeness requirement in d)  $\Rightarrow$  a) of Theorem 16 has been removed [33].
3. Further investigation of the influence of small cardinals on Michael's problem can be found in [1].
4. A minor variant of Moore's  $L$ -space constructed by Y. Peng has non-Lindelöf square [29].

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Leandro F. Aurichi  
 Instituto de Ciências Matemáticas de Computação (ICMC-USP),  
 Universidade De São Paulo, São Carlos, SP  
 CEP 13566-590 - Brazil  
*e-mail address:* aurichi@icmc.usp.br

Franklin D. Tall  
 Department of Mathematics  
 University of Toronto  
 Toronto, Ontario  
 M5S 2E4  
 CANADA  
*e-mail address:* f.tall@utoronto.ca