Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or D

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This paper is dedicated to Ken Kunen, who, in addition to his own ground-breaking research, has been insightfully guiding graduate students for 40 years. The dedication from my doctoral dissertation is still apt: he provides clear answers to murky questions.

Abstract

We discuss relationships in Lindelöf spaces among the properties "Menger", "Hurewicz", "Alster", "productive", and "D".

This note is a continuation of [13]. The question of what additional assumptions ensure that the product of two Lindelöf spaces is Lindelöf is natural and well-studied. See e.g., [28], [30], [2], [3], [4], [5], [6], [32], [33].

D-spaces were introduced in [20].

Definition. *A space X is* **D** *if for every* ***neighbourhood assignment*** $\{V_x\}_{x\in X}$*, i.e. each V_x is an open set containing x, there is a closed discrete $Y \subseteq X$ such that $\{V_x\}_{x\in Y}$ covers X. Y is called a* ***kernel*** *of the neighbourhood assignment.*

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The question raised in [20] of whether every Lindelöf space is a D-space has been surveyed in [22]. It has recently been the subject of much research. In [12], Aurichi established many connections among the D property, topological games, and selection properties. In this paper, we examine connections among selection principles, the D property, and preservation of Lindelöfness under products.

Definition [16]**.** *A space X is **productively Lindelöf** if $X \times Y$ is Lindelöf for any Lindelöf space Y.*

Definition [3],[16]**.** *A space is **Alster** if every cover by G_δ sets that covers each compact set finitely includes a countable subcover.*

Lemma 1 [3]**.** *Every Alster space is productively Lindelöf; CH implies every productively Lindelöf T_3 space of weight $\leq \aleph_1$ is Alster.*

The following definitions are equivalent to other ones for these properties:

Definition. *A space is **Rothberger (Menger)** if for each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open covers, (such that each finite union of elements of $\mathcal{U}_n$ is a member of $\mathcal{U}_n$,) there are $U_n \in \mathcal{U}_n$, $n < \omega$, such that $\bigcup_{n<\omega} U_n$ is an open cover.*

Lemma 2 [12]**.** *Every Menger space is D.*

Lemma 3 [13]**.** *Every Alster space is Menger.*

Corollary 4 [13]**.** *Every Alster space is D.*

There are various definitions of the *Hurewicz* property. It will be convenient to use the one in [26].

Definition. *A γ-**cover** of a space is a countably infinite open cover such that each point is in all but finitely many members of the cover. A space is **Hurewicz** if given a sequence $\{\mathcal{U}_n : n \in \omega\}$ of γ-covers, there is for each n a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$, such that either $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a γ-cover, or else for some n, $\bigcup \mathcal{V}_n$ is a cover.*

For sets of reals, Hurewicz fits strictly between σ-compact and Menger — see e.g. [36].

Lemma 5 [26]**.** *A set X of real numbers is Hurewicz if and only if for every G_δ G including X, there is an F_σ F such that $X \subseteq F \subseteq G$.*

The characterization of Hurewicz sets of reals given in Lemma 5 can be extended to arbitrary Hurewicz spaces by repeating the proof in [26] in a more general context.

Theorem 6. *A Lindelöf T_3 space X is Hurewicz if and only if for every Čech-complete $G \supseteq X$, there is a σ-compact F such that $X \subseteq F \subseteq G$.*

Proof. Let $X \subseteq G = \bigcap_{n<\omega} G_n$, where G_n are open and dense in a compact K. Choose for each $x \in X$ an open U_x^n such that $x \in U_x^n \subseteq \overline{U_x^n} \subseteq G_n$, where the closure is taken in K. Since Hurewicz implies Lindelöf, choose a countable subcover $\{U_{x_n^j}^n : x \in X\}$. For each n and k, define $U_k^n = \bigcup_{j\leq k} U_{x_n^j}^n$. Then $\mathcal{U}_n = \{U_k^n : k \in \omega\}$ is a γ-cover of X such that for each k, $\overline{U_k^n} \subseteq G_n$. Applying Hurewicz, in the non-trivial case, for each n choose a k_n such that $\{U_{k_n}^n : n \in \omega\}$ is a γ-cover of X. For each n, let $F_n = \bigcap\{\overline{U_{k_m}^m} : m \geq n\}$. Then $F_n \subseteq G$, so $X \subseteq \bigcup_{n<\omega} F_n \subseteq G$. $\bigcup_{n<\omega} F_n$ is σ-compact, as required. In the trivial case, we in fact get X included in a compact subset of G.

Conversely, let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of countable open covers of X such that for each $n \in \omega$ and each $x \in X$, $\{U \in \mathcal{U}_n : x \notin U\}$ is finite. For each n and each $U \in \mathcal{U}_n$, pick a $U'(n)$ open in βX such that $U = U'(n) \cap X$. Let $\mathcal{U}'_n = \{U'(n) : U \in U_n\}$. Let $\mathcal{U}^*_n = \bigcup\{U'(n) : U \in \mathcal{U}_n\}$. Then $\bigcap\{U^*_n : n \in \omega\}$ is a Čech-complete subspace of βX including X. Take compact F_n, $n \in \omega$, such that $X \subseteq \bigcup\{F_n : n \in \omega\} \subseteq \bigcap_{n<\omega} U^*_n$. Then $\bigcup\{F_m : m < n\}$ is compact, and so is included, for each n, in some finite $\mathcal{W}'_n \in [\mathcal{U}'_n]^{<\omega}$. Let $\mathcal{W} = \{U : U'(n) \in \mathcal{W}'_n\}$ If for no n does $\bigcup \mathcal{W}_n = X$, then $\{\bigcup \mathcal{W}_n : n \in \omega\}$ is infinite, and each $x \in X$ is in all but finitely many $\bigcup \mathcal{W}_n$. □

After completing an earlier draft of this note, I found that Banakh and Zdomskyy [14] had recently proved this, but only for separable metrizable spaces. Again, their proof can be easily modified to yield the general case. Using this characterization, we can improve Lemma 3 (since Hurewicz implies Menger) and answer a question posed to us by Marion Scheepers by proving:

Theorem 7. *Every T_3 Alster space is Hurewicz.*

First, a definition:

Definition [9]**.** *A space is of* ***countable type*** *if every compact set is included in a compact set of countable character.*

Clearly in a space of countable type, every compact set is included in a compact G_δ. Čech-complete spaces, metrizable spaces, and indeed p-spaces in the sense of Arhangel'skiĭ [8] are of countable type.

Proof of Theorem 7. Let Y be a Čech-complete space including an Alster X. Then every compact subspace of X is included in a compact G_δ in Y. Consider the G_δ cover of X obtained from the intersection of those compact G_δ's with X. It covers each compact subspace of X by one element of the cover, so since X is Alster, countably many of those elements cover X. But each of these is the intersection of a compact G_δ of Y with X. Thus X is included in a σ-compact subspace of Y, as required. □

Notice that this characterization of Hurewicz spaces easily yields that Hurewicz is a perfect invariant:

Theorem 8. *For T_3 spaces, perfect images of Hurewicz spaces are Hurewicz, as are perfect preimages.*

Proof. These follow from σ-compactness and Čech-completeness being perfect invariants for $T_{3\frac{1}{2}}$ spaces. See [23] for a proof of the latter. □

Question 14 of Alster [5] asks whether, under MA, *if for every separable metrizable Čech-complete Y including X there is a σ-compact Z in Y including X, then X must be the union of less than continuum compact sets.* We can answer this negatively by using Theorem 6. It merely suffices to let X be a Hurewicz set of reals which is not σ-compact [26] and to assume CH (which implies MA). By Theorem 6, X satisfies Alster's requirements, yet it is not σ-compact.

Alster [3] proved that Alster spaces in which compact sets are G_δ are σ-compact. We can weaken that condition:

Theorem 9. *If X is Alster and every compact subset of X is included in a compact G_δ, then X is σ-compact.*

Proof. Cover X by compact G_δ sets, taking for each compact set one covering it. The resulting cover then has a countable subcover, so X is σ-compact. □

Corollary 10. *Every Alster space of countable type, e.g. every Alster Čech-complete space, is σ-compact.*

Definition. *Partially order* ${}^{\omega}\omega$ *by* $f \leq_* g$ *if* $f(n) \leq g(n)$ *for all but finitely many* n*.* $\mathfrak{b}$ *is the least cardinal such that for every* $\mathcal{F} \in {}^{\omega}\omega$ *of size* $< \mathfrak{b}$*, there is a* $g \in {}^{\omega}\omega$ *such that for each* $f \in \mathcal{F}$*,* $f \leq_* g$*.* $\mathfrak{d}$ *is the least cardinal* δ *such that there is a family* $\mathcal{F}$ *of size* $\delta \in {}^{\omega}\omega$ *such that for every* $f \in {}^{\omega}\omega$*, there is a* $g \in \mathcal{F}$*, such that* $f \leq_* g$*.* $\mathbf{cov}(\mathcal{M})$ *is the least cardinal* δ *such that* ${}^{\omega}\omega$*, identified with* $\mathbb{P}$*, is the union of* δ *nowhere dense sets.* $\mathbf{add}(\mathcal{M})$ *is the least cardinal* κ *such that the union of fewer than* κ *first category subsets of* $\mathbb{P}$ *is first category. A* ***λ-scale*** *is a subset* S *of* ${}^{\omega}\omega$ *such that* $<_*$ *(i.e.,* $\leq_*$*, but not for all but finitely many* n *equal) well-orders* S *and each* $f \in {}^{\omega}\omega$ *is less than some member of* S*.*

The question of whether productively Lindelöf spaces are Alster was asked in [3] and [16]. σ-compact spaces are Alster [3]; Alster metrizable spaces are σ-compact [3]. Thus for metrizable spaces, the question is whether productive Lindelöfness implies σ-compactness. Michael [28] obtained this from CH; it was proved explicitly in [2]; this was slightly improved in [13] to get:

Lemma 11. $\mathfrak{d} = \aleph_1$ *implies productively Lindelöf metrizable spaces are* σ*-compact.*

The Menger property is a weakening of σ-compactness. In completely metrizable spaces, Menger is equivalent to σ-compact [25]. However, there are Menger sets of reals of size $\aleph_1$ which are not σ-compact [19], [26], [36]. We will prove:

Theorem 12. *Productively Lindelöf metrizable spaces which are the union of* $\aleph_1$ *compact sets are Menger.*

Proof. We have already dealt with the case when $\mathfrak{d} = \aleph_1$; now consider the case when $\mathfrak{d} > \aleph_1$.

This follows from:

Lemma 13. *Lindelöf spaces which are the union of* $< \mathfrak{d}$ *compact sets are Menger.*

This is an improvement due to Ofelia Alas over previous results which assumed separation axioms and assumed size $< \mathfrak{d}$. I am grateful to her for suggesting I include Lemma 13 in this paper.

Proof. Let $X = \bigcup\{K_\alpha : \alpha < \kappa\}$, where each K_α is compact, and $\kappa < \mathfrak{d}$. Fix a countably infinite sequence of open covers $\{\mathcal{U}_n : n < \omega\}$, $\mathcal{U}_n = \{U_{n,m}\}_{m<\omega}$. For each $\alpha < \kappa$, define $f_\alpha : \omega \to \omega$ such that $K_\alpha \subseteq \bigcup\{U_{n,m} : m \leq f_\alpha(n)\}$. Since $\kappa < \mathfrak{d}$, there is a $g : \omega \to \omega$ such that for no α is $g \leq_* f_\alpha$. For each $n < \omega$, define $\mathcal{V}_n = \{U_{n,0}, \ldots, U_{n,g(n)}\}$. Then $\bigcup\{\mathcal{V}_n : n < \omega\}$ covers, since if $x \in X$, there is an $\alpha < \kappa$ such that $x \in K_\alpha$ and an $n < \omega$ such that $g(n) > f_\alpha(n)$ and so $x \in \bigcup\{\mathcal{V}_n : n < \omega\}$. □

Notice that Theorem 12 can be strengthened:

Corollary 14. *Finite powers of productively Lindelöf metrizable spaces which are the union of $\aleph_1$ compact sets are Menger.*

Proof. Productive Lindelöfness is preserved by finite products [16]. □

The condition that finite powers of X are Menger is equivalent to $C_p(X)$ having *countable fan tightness* [10].

Assuming a claim of Alster [3], Theorem 12 could be improved to:

***Theorem 15.** *Suppose $w(X) \leq \aleph_1$, $|X| \leq \aleph_1$, compact subsets of X are G_δ's, and X is productively Lindelöf T_3. Then X is Menger and hence D.*

Proof. The case for $\mathfrak{d} > \aleph_1$ is as before. For $\mathfrak{d} = \aleph_1$, Lemma 11 would give it to us, assuming Alster's claim [3] that:

***Lemma 16.** *If $w(X) \leq \aleph_1$, X is productively Lindelöf, and every compact subset of X is G_δ, then X is σ-compact if and only if every metrizable productively Lindelöf space is σ-compact.*

Unfortunately, I cannot follow his hint for a proof. It is hard to find spaces satisfying the hypotheses of *Theorem 15, if $\aleph_1 < 2^{\aleph_0}$. They cannot be Alster, since Alster spaces in which compact sets are G_δ are σ-compact. No example of a productively Lindelöf space which is not Alster is known, even under additional set-theoretic axioms.

From Lemma 1 and Theorem 9, we get that:

Corollary 17. *CH implies every productively Lindelöf T_3 space of countable type and weight $\leq \aleph_1$ is σ-compact.*

Corollary 17 could presumably be used to improve *Theorem 15, relaxing "every compact set is a G_δ" to "every compact set is included in a compact G_δ". The reason is that the only use of "every compact set is a G_δ" in the "proof" of *Lemma 16 would likely be to get that an Alster space is σ-compact.

Thus we would have:

***Corollary 18.** *Suppose $|X| \leq \aleph_1$ and X is productively Lindelöf T_3 and of countable type. Then X is Menger and hence D.*

Proof. For a space X of countable type, $w(X) \leq |X|$. To see this — which is undoubtedly due to Arhangel'skiĭ— note that it suffices to show that $\chi(X) \leq |X|$. Each point p is contained in a compact set K of size $\leq |X|$. For compact spaces K, we know $w(K) \leq |K|$. Thus $\chi(p, X) \leq \chi(p, K) \cdot \chi(K, X)$.

Since we may take a K such that $\chi(K, X) \leq \aleph_0$, the result follows. □

Alster has now informed me that his hint for *Lemma 16 is incorrect, and that he does not know whether *Lemma 16 is correct.

In contrast to Theorem 12 and *Theorem 15, it is easy to find an uncountable space satisfying the hypotheses of *Corollary 18, e.g. $\omega_1 + 1$.

Alster [5] asked whether MA was sufficient to imply that productively Lindelöf metrizable spaces are σ-compact. We go part way toward answering his question:

Theorem 19. *$add(\mathcal{M}) = \mathfrak{c}$ implies (finite products of) productively Lindelöf metrizable spaces are Hurewicz.*

As with Menger, the "finite powers of" follows from the prima facie weaker statement, since finite products of productively Lindelöf spaces are productively Lindelöf. Before proving the weaker statement, we set out a number of useful lemmas. First, a couple of classical results, and then some sharpenings of Lemma 5.

Lemma 20 (folklore, see e.g. #208, p.347 of [11])**.** *Every separable metrizable space is a perfect image of a 0-dimensional separable metrizable space.*

Lemma 21 [27, I.7.8]**.** *Every 0-dimensional separable metrizable Čech-complete space is homeomorphic to a closed subspace of $\mathbb{P}$, the space of irrationals, considered as ω^ω.*

Lemma 22. *If $X \subseteq \mathbb{P}$, then X is Hurewicz if and only if every G_δ subset of $\mathbb{P}$ including X includes an F_σ subset of $\mathbb{P}$ including X.*

Proof. For the non-trivial direction, suppose X is Hurewicz and that G is a G_δ subset of $\mathbb{P}$ including X. G is completely metrizable and 0-dimensional and so is homeomorphic to a closed subspace of $\mathbb{P}$. Without loss of generality we may therefore assume it is closed in $\mathbb{P}$. G is also G_δ in $\mathbb{R}$, so by Lemma 5 it includes an F_σ subset F of $\mathbb{R}$ including X. But then F is σ-compact and included in $\mathbb{P}$, so F_σ in $\mathbb{P}$, as required. □

Lemma 23. *A 0-dimensional subset X of $\mathbb{R}$ is Hurewicz if and only if every homeomorph of X included in $\mathbb{P}$ is included in a σ-compact subspace of $\mathbb{P}$.*

Proof. Suppose X is not Hurewicz. Then there is a G_δ subspace G of $\mathbb{P}$ not including any σ-compact subspace of $\mathbb{P}$ including X. G is homeomorphic to a closed subspace of $\mathbb{P}$, say $h''G$. Then $h''X$ is not included in any σ-compact subspace S of $\mathbb{P}$, for if it were, $h''G \cap S$ would be σ-compact, as would $h^{-1}(h''G \cap S)$. □

As we saw earlier, "Hurewicz" is a perfect invariant for $T_{3\frac{1}{2}}$ spaces. Also observe:

Lemma 24. *"Productively Lindelöf" is a perfect invariant.*

Proof. Suppose $\pi : X \to Y$ is perfect and onto. Let Z be Lindelöf. $\pi \times id_Z : X \times Z \to Y \times Z$ is perfect, so $Y \times Z$ is Lindelöf if and only if $X \times Z$ is. □

Lemma 25 [29, Theorem 1.2]**.** *$add(\mathcal{M}) = \min\{cov(\mathcal{M}), \mathfrak{b}\}$ and hence assuming $cov(\mathcal{M}) = \mathfrak{c}$ plus there is a $\mathfrak{c}$-scale is equivalent to assuming $add(\mathcal{M}) = \mathfrak{c}$.*

I thank Marion Scheepers for pointing this out to me.

Lemma 26 [4]**.** *$cov(\mathcal{M}) = \mathfrak{c}$ implies that if a compact metrizable space is the union of $< 2^{\aleph_0}$ compact sets, it is actually the union of countably many of them.*

Note. We defined $cov(\mathcal{M})$ for the meager ideal in $\mathbb{P}$, but this is an upper bound for the covering number of the meager ideal in an arbitrary compact metric space, which is what Alster uses.

In the proof following Remark 16 in [5], Alster in effect shows that *if MA implies every Hurewicz metrizable space is the union of fewer than $2^{\aleph_0}$ compact sets* [which it doesn't, because that assertion is false], *then productively Lindelöf metrizable spaces are σ-compact.* I don't know if that conclusion is true. Despite the false premise, a piece of Alster's proof, combined with the lemmas above, will enable us to prove Theorem 19.

Proof. Assume MA and that X is metrizable, but not Hurewicz. We will show that X is not productively Lindelöf. First, by Theorem 8 and Lemmas 20 and 21 we may assume without loss of generality that $X \subseteq \mathbb{P}$. Following Alster, we divide into cases depending on whether or not X is the union of fewer than $2^{\aleph_0}$ compact sets. In the case where X is not such a union, let Y be a metric compactification of $\mathbb{P}$ and let $X_\alpha = \{p \in \mathbb{P} : p \leq_* x_\alpha\}$, where $\{x_\alpha : \alpha < \mathfrak{c}\}$ is a $\mathfrak{c}$-scale, and let $X_\mathfrak{c}$ be the set Y. The topology on $X_\mathfrak{c}$ is generated by sets of the form $(X_{\alpha_2} - X_{\alpha_1}) \cap H$, where $-1 \leq \alpha_1 < \alpha_2 \leq \mathfrak{c}$, H is open in Y and $X_{-1} = \emptyset$. In [4], using Lemma 26, Alster proves $X_\mathfrak{c}$ is Lindelöf. In [5], he proves $X \times X_\mathfrak{c}$ is not Lindelöf, claiming and using that without loss of generality there is no σ-compact subspace of $\mathbb{P}$ including X. We have proved that above as Lemma 23.

For the other case, we can simply prove in ZFC that:

Lemma 27. *A Lindelöf space which is the union of fewer than $\mathfrak{b}$ compact subspaces is Hurewicz.*

Proof. Again I thank Ofelia Alas for suggesting I include her proof, which is shorter than mine and does not require separation axioms. Follow the notation of the proof of Lemma 13, this time with $\kappa < \mathfrak{b}$ and with the $\mathcal{U}_n$'s being γ-covers. Then there is a $g : \omega \to \omega$ such that for each $\alpha < \kappa$, for all but finitely many n, $f_\alpha(n) \leq g(n)$. Defining $\mathcal{V}_n$ as before, given any $x \in X$, x is in some K_α and $f_\alpha \leq_* g$. Then for all but finitely many n, $x \in \bigcup \mathcal{V}_n$. □

We don't know whether in ZFC productively Lindelöf (metrizable) spaces are Alster, Hurewicz, or Menger. Moore's L-space [31] is Hurewicz [35], but its square is not Lindelöf. A Hurewicz set of reals which is not σ-compact will, under CH, by Lemma 1 not be productively Lindelöf. We don't know whether CH is necessary here.

Thus, finishing the proof of Theorem 19, in either case, X is not productively Lindelöf. □

After this paper was almost completed, I ran across yet another paper of Alster: [6], and found that most of Theorem 19 had been anticipated by him, using an equivalent version of "Hurewicz". He also mentions in a note added in proof that R. Pol has answered his Question 1, but gives no details. That question is equivalent to Question 14 of [5] which we answered above. No doubt Pol answered it the same way.

Using an idea in [6], I can now improve a result in [13]. There we proved that *every completely metrizable productively Lindelöf space is σ-compact if and only if there is a* **Michael space**, i.e. a Lindelöf space X such that $X \times \mathbb{P}$ is not Lindelöf. We can now prove:

Theorem 28. *Every analytic, metrizable, productively Lindelöf space is σ-compact if and only if there is a Michael space.*

Proof. Suppose X is analytic, metrizable, productively Lindelöf, and not σ-compact. Without loss of generality, we may as usual assume that X is 0-dimensional and hence included in $\mathbb{P}$ and hence in $\mathbb{R}$. The one non-obvious point here is that perfect pre-images of analytic spaces are analytic — see [27, p. 196]. We then apply Hurewicz's theorem: 21.18 of [27] to conclude that an analytic set of reals which is not σ-compact includes a closed copy of $\mathbb{P}$. But then, if there is a Michael space, such a set X is not productively Lindelöf. □

I conjecture that Theorem 28 can be improved to "projective" rather than "analytic" assuming **PD**, *The Axiom of Projective Determinacy.* As a partial step, we prove:

Theorem 29. *PD implies that every projective, productively Lindelöf space is Hurewicz if and only if there is a Michael space.*

Proof. We need:

Lemma 30 [27, 38.18]**.** *PD implies a projective subset of a separable, completely metrizable space either includes a closed copy of $\mathbb{P}$ or is included in a σ-compact subspace.*

Without loss of generality, we may assume our projective, metrizable, productively Lindelöf space A is included in $\mathbb{P}$. If it were to include a closed copy of $\mathbb{P}$, then it would not be productively Lindelöf, if there is a Michael space. Suppose instead that A is included in a σ-compact subspace of $\mathbb{P}$.

Suppose A were not Hurewicz. Then some homeomorph of A, $h''A$, would not be included in any σ-compact subspace of $\mathbb{P}$. But $h''A$ is projective, so must then include a closed copy of $\mathbb{P}$. Then so does A, so it is not productively Lindelöf if there is a Michael space. □

It is a natural question whether the product of two Lindelöf D-spaces is D. Note that finite products of Alster spaces are D, because they are in fact Alster [15], [16]. There are several pairs of spaces X, Y in the literature that are Lindelöf with non-Lindelöf product, yet that product has countable extent, and so is not D. It is not so clear whether these spaces are D; they can be made D without changing their other relevant properties by adding $\aleph_1$ Cohen reals. This is detailed below. In fact, I have been informed by Lucia Junqueira that a space in [1, Theorem 2.11] constructed from an Eric line is a Lindelof D-space whose product with a particular separable metrizable (hence Lindelöf D) space is not D. The Cohen real adjunction still has some interest, for it produces Menger (indeed Rothberger) spaces with square non-D. Aurichi's game-theoretic proof that Menger spaces are D [12] could lead one to conjecture that finite powers of a Menger space are D, but consistently, this is not true. It is not clear whether the space in [1] can be improved to be Menger.

Theorem 31. *It is consistent that there are Rothberger spaces X, Y such that $X \times Y$ is not D, indeed is linearly Lindelöf but not Lindelöf.*

Proof. Moore [30] proves that $2^{\aleph_0} > \aleph_\omega$ implies there are Lindelöf spaces X, Y such that $X \times Y$ is linearly Lindelöf but not Lindelöf. Linearly Lindelöf spaces are Lindelöf if they are D, since they have no uncountable closed discrete subspaces. Adding $\aleph_1$ Cohen reals will make X and Y Rothberger and hence D [35]. It will also keep $X \times Y$ not Lindelöf [21]. It thus only remains to prove that $X \times Y$ remains linearly Lindelöf. To see this, we need to show that every open cover in the extension with size non-ω-cofinal has a countable subcover. This is done by exactly the same argument as the one in [24] for proving Lindelöfness is preserved by adding Cohen reals. □

Corollary 32. *It is consistent that there is a Rothberger space R such that R^2 is not D.*

Proof. As noted by participants in the Toronto Set Theory Seminar, it suffices to consider the topological sum $R = X \oplus Y$ of the spaces of Theorem 31. Then R is Rothberger, but R^2 includes $X \times Y$ as a closed subspace, so it is not D. □

We can vary the proof of Theorem 31 by using another example; we lose "linearly Lindelöf", but the continuum need not be so large:

Theorem 33. *It is consistent that* $2^{\aleph_0} = \aleph_2$ *and there is a Rothberger space* R *such that* R^2 *is not* D.

Proof. Alster [4] proved:

Lemma 34. $MA + 2^{\aleph_0} > \aleph_1$ *implies that if* X *is Lindelöf, then* $X \times \mathbb{P}$ *has countable extent.*

Lemma 35. MA *implies there is a Lindelöf space* X *such that* $X \times \mathbb{P}$ *is not Lindelöf.*

It follows that under $MA + 2^{\aleph_0} > \aleph_1$, Alster's example $X \times \mathbb{P}$ is not D. Adding $\aleph_1$ Cohen reals will, as before, make X Rothberger and keep $X \times \mathbb{P}$ not Lindelöf.

The Cohen reals will preserve countable extent by the usual argument, since this translates as "every open cover of size $\aleph_1$ has a countable subcover". □

A more difficult example of two Lindelöf spaces with product not Lindelöf is in [32]. According to [7] the product has countable extent. Thus by adding $\aleph_1$ Cohen reals over a suitable ground model, we can get a Rothberger space with non-D square and whatever cardinal arithmetic we like.

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