

PFA(S)[S] and the Arhangel'skiĭ-Tall problem

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Abstract

We discuss the Arhangel'skiĭ-Tall problem and related questions in models obtained by forcing with a coherent Souslin tree.

1 Introduction

Around 1965, A.V. Arhangel'skiĭ proved:

Proposition 1. *Every locally compact, perfectly normal, metacompact space is paracompact.*

In response to my question to him in Prague in 1971 as to whether this was true, he responded that he had proved it, but his mentor, P. S. Alexandrov, had not thought it worth publishing! He subsequently published it in [2]. Neither of us could answer the question of what happened if the “closed sets are G_δ 's” requirement was dropped. I raised this in [32] and it became known as the “Arhangel'skiĭ-Tall” problem. A partial solution was achieved in [37], where S. Watson proved:

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Proposition 2. *$V = L$ implies every locally compact normal metalindelöf space is paracompact.*

Then G. Gruenhage and P. Koszmider [16] proved:

Proposition 3. *If ZFC is consistent, it is consistent with the existence of a locally compact, normal, metacompact space which is not paracompact.*

In 2003, using results announced by S. Todorcevic (which have now been proved in [19] plus [10]), P. Larson and I [22] proved:

Theorem 4. *If the existence of a supercompact cardinal is consistent with ZFC, so is the assertion that every locally compact, perfectly normal space is paracompact.*

The question remained as to whether one could obtain the paracompactness of locally compact normal metacompact spaces as well as the conclusion of Theorem 4 in the same model, i.e. could we change Arhangel'skiĭ's "and" to an "or"? That is what we shall do here, subject to the same large cardinal assumption as in Theorem 4. We conjecture that that assumption can be eliminated.

Theorem 5. *If the existence of a supercompact cardinal is consistent with ZFC, so is the assertion that every locally compact normal space that either is metalindelöf or has all closed sets G_δ 's is paracompact.*

2 PFA(S)[S]

The model that we use first is the same one as for Theorem 4. I use the convention that "PFA(S)[S] implies Φ " stands for the assertion that in any model constructed by starting with a coherent Souslin tree S , forcing to obtain PFA(S), i.e. PFA restricted to proper posets preserving S , and then forcing with S , Φ holds. We analogously use "MA $_{\omega_1}$ (S)[S]" . For a discussion of such models and the definition of *coherent*, see [23]. The model of Theorem 4 is a model constructed in that fashion, but over a particular ground model. We shall later discuss modifications of that model.

That the model of Theorem 4 suffices to prove Theorem 5 follows immediately from:

Theorem 6 [30]. *PFA(S)[S] implies locally compact normal spaces are \aleph_1 -collectionwise Hausdorff.*

Lemma 7 [17]. *Locally compact normal \aleph_1 -collectionwise Hausdorff metalindelöf spaces are paracompact.*

□

Gruenhage and Kozmider proved in [17] that:

Proposition 8. *MA_{ω_1} implies every locally compact, normal, metalindelöf space is paracompact.*

Their only use of MA_{ω_1} was to prove the following proposition:

Proposition 9. *Assume MA_{ω_1} . Let $\{B_\alpha : \alpha < \omega_1\}$ be a collection of sets such that whenever $\{F_\alpha : \alpha < \omega_1\}$ is a disjoint collection of finite subsets of ω_1 , $\{\bigcup B_\beta : \beta \in F_\alpha, \alpha < \omega_1\}$ is not centered. Let $\{Y_\alpha : \alpha < \omega_1\}$ be a collection of countable sets such that $|Y_\alpha - \bigcup\{B_\beta : \beta \in F\}| = \aleph_0$, for every finite $F \subseteq \omega_1 - \{\alpha\}$. Then $\omega_1 = \bigcup_{n < \omega} A_n$, where for each $n \in \omega$ and $\alpha \in \omega_1$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in A_n - \{\alpha\}\}| = \aleph_0$.*

In fact, analyzing their use of Proposition 9 in their proof, we observe that they only needed that each stationary $S \subseteq \omega_1$ included a stationary T such that for every $\alpha \in T$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in T - \{\alpha\}\}| = \aleph_0$. This follows from there being a closed unbounded C such that for $\alpha \in C$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in C - \{\alpha\}\}| = \aleph_0$. We conjecture this follows from PFA(S)[S].

Watson [38] constructed a locally compact, perfectly normal, metalindelöf, non-paracompact space from $MA_{\omega_1}(\sigma\text{-centered})$ plus the existence of a Souslin tree. It is certainly consistent that there are no locally compact, perfectly normal, metalindelöf spaces that are not paracompact. It follows from Proposition 8, and in fact I showed that MA_{ω_1} implied there weren't any a long time ago in [32].

3 Weakening the model of Theorem 4

One wonders whether all of the requirements of the model of Theorem 4 are necessary. Whether large cardinals are necessary has not yet been investigated. I conjecture that they are not, except possibly for an inaccessible. Avoiding that issue, two others remain:

1. Is the preliminary forcing used in [22] before forcing $\text{PFA}(S)[S]$ necessary?
2. Do we just need $\text{PFA}(S)[S]$, or do we need a model of $\text{PFA}(S)[S]$ constructed by the usual iteration, i.e. following the usual proof of the consistency of PFA, but using only those partial orders that preserve S [25]?

We can answer the first question negatively; our particular answer however requires the second alternative for the second question. The preliminary forcing in [22] — adding λ^+ Cohen subsets of λ for every regular $\lambda \geq$ a supercompact κ — was done so as to assure we could get full collectionwise Hausdorffness from the \aleph_1 -collectionwise Hausdorffness provided by the Souslin tree forcing. An old consistency result of Shelah [28] recast as a proof from a reflection axiom [13], [8], [14], tells us that under such an axiom, locally separable, first countable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff. However, such reflection axioms do not follow from $\text{PFA}(S)[S]$, but require a stronger principle holding in the usual iteration model for $\text{PFA}(S)[S]$. Now for the details.

First of all, the relevance of “local separability” is:

Lemma 10. *If every first countable, hereditarily Lindelöf, regular space is hereditarily separable, then locally compact perfectly normal spaces are locally separable.*

Proof. To see this, note that locally compact, perfectly normal spaces are first countable. Next, note:

Lemma 11 [24]. *$\text{MA}_{\omega_1}(S)[S]$ implies every first countable, hereditarily Lindelöf, regular space is hereditarily separable.*

□

$\text{MA}_{\omega_1}(S)[S]$ of course follows from $\text{PFA}(S)[S]$. There are two reflection axioms in the literature we want to focus on, but we will not actually need their complicated definitions. “Axiom R ” was introduced by Fleissner [13], who proved it implied *locally separable, first countable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff*. In [15], the authors interpolated a new axiom, *FRP*, obtaining:

Lemma 12 [14], [15]. *Axiom R implies FRP, which implies every locally separable, first countable, \aleph_1 -collectionwise Hausdorff space is collectionwise Hausdorff.*

Definition [5]. $\text{MA}_{\omega_1}(\text{countably closed}, \kappa)$ is the assertion that if P is a countably closed partial order, \mathcal{D} is a family of at most \aleph_1 dense subsets of P , and $\{S_\alpha : \alpha < \kappa\}$ is a family of cardinality κ of P -terms, each forced by every condition in P to denote a stationary subset of ω_1 , then there is a \mathcal{D} -generic filter G on P so that for every $\alpha < \kappa$, $S_\alpha(G)$ is stationary, where:

$$S_\alpha(G) = \{\beta < \omega_1 : (\exists p \in G)p \Vdash \check{\beta} \in \dot{S}_\alpha\}.$$

$\text{MA}_{\omega_1}(\text{proper}, \kappa)$ is defined analogously. Baumgartner [4] denotes $\text{MA}_{\omega_1}(\text{proper}, \aleph_1)$ by PFA^+ ; some authors use PFA^{++} for “ $\text{MA}_{\omega_1}(\text{proper}, \aleph_1)$ ” and “ PFA^+ ” for “ $\text{MA}_{\omega_1}(\text{proper}, 1)$ ”. We shall use Baumgartner’s notation.

It is known that:

Lemma 13 [4]. PFA^+ holds in the usual iteration model for PFA .

Lemma 14 [5]. $\text{MA}_{\omega_1}(\text{countably closed}, 1)$ implies Axiom R.

Since countably closed partial orders preserve Souslin trees, we see that $\text{MA}_{\omega_1}(\text{countably closed}, 1)(S)$ also implies Axiom R and so PFA^+ does as well. It is not known whether $\text{MA}_{\omega_1}(\text{proper}, 1)(S)[S]$ implies Axiom R, but $\text{PFA}^+(S)[S]$ implies Axiom R [21].

For *FRP*, there is a less specialized result:

Lemma 15 [14]. *FRP is preserved by countable chain condition forcing.*

Corollary 16. $\text{MA}_{\omega_1}(\text{countably closed}, 1)(S)[S]$ implies *FRP*.

We cannot, however, drop the one remaining stationary set:

Theorem 17. $\text{PFA}(S)[S]$ does not imply first countable, locally separable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff.

Proof. Beaudoin notes that he and M. Magidor have independently shown that PFA is consistent with the existence of a non-reflecting stationary $E \subseteq \{\alpha < \omega_2 : cf(\alpha) = \omega\}$. Such a set is well-known to yield a ladder system space which is first countable, locally separable, \aleph_1 -collectionwise Hausdorff

but not collectionwise Hausdorff [11]. It thus only remains to show that such a space is preserved by the adjunction of a Souslin tree. The first two properties are “basis properties” and are clearly preserved. For the space to become collectionwise Hausdorff, the stationarity of E would have to be destroyed, which countable chain condition forcing can’t do. It remains to show that \aleph_1 -collectionwise Hausdorffness is preserved. The reason is that, by a standard argument, every subset Y of size \aleph_1 of a ground model set X in a c.c.c. extension is included in a ground model subset Z of X of size \aleph_1 . The ground model separation of Z then restricts to a separation of Y . \square

Note, however, that we have not proved that $\text{PFA}(S)[S]$ does not imply locally compact, perfectly normal spaces are collectionwise Hausdorff. We conjecture that this can be accomplished by proving that a finite condition variant of the partial order Shelah uses in [28] to force the ladder system space mentioned above to be normal is proper and preserves S . Ladder system spaces are locally compact Moore spaces and hence have closed sets G_δ , so that would suffice.

In addition to needing that first countable, hereditarily Lindelöf, regular spaces are hereditarily separable, and that locally compact, perfectly normal spaces are collectionwise Hausdorff, the proof in [22] of the consistency of locally compact, perfectly normal spaces being paracompact needed:

Lemma 18 [33],[34]. $\text{PFA}(S)[S]$ implies Σ .

Definition. *Balogh’s Σ* is the assertion that if Y is a subset of size \aleph_1 of a compact, countably tight space X , and there is a family \mathcal{V} of \aleph_1 open sets covering Y such that for every $V \in \mathcal{V}$ there is an open $U_V \subseteq X$ such that $\overline{V} \subseteq U_V$ and $U_V \cap Y$ is countable, then Y is σ -discrete.

The status of Todorćević’s proof is as follows. He announced the result in a seminar in Toronto in 2002 and in a lecture in Prague in 2006. He sketched the proof of the hardest part — that $\text{PFA}(S)[S]$ implies that compact, countably tight spaces are sequential — in his lectures in Erice in 2008 [34]. He sketched a proof of a weaker version of Balogh’s Σ restricted to compact sequential spaces in notes in 2002 [33]. A proof that avoids the necessity for proving compact countably tight spaces are sequential now exists in the union of [19] plus [10].

4 $\text{MA}_{\omega_1}(S)[S]$ does not imply there are no first countable S -spaces

The following material deals with a question analogous to what we have considered so far: does a result proved to hold in a particular model of $\text{MA}_{\omega_1}(S)[S]$ actually follow from $\text{MA}_{\omega_1}(S)[S]$?

A key unresolved question is *whether* $\text{PFA}(S)[S]$ implies there are no first countable S -spaces. I had incorrectly claimed this at a couple of conferences in 2006. If this is true, it would follow that $\text{PFA}(S)[S]$ implies there are no first countable, hereditarily normal, separable Dowker spaces. This is because of the following proposition from [22]:

Proposition 19. $\text{MA}_{\omega_1}(S)[S]$ implies first countable hereditarily normal spaces satisfying the countable chain condition are hereditarily separable.

We shall now show that $\text{MA}_{\omega_1}(S)[S]$ is not sufficient to prove there are no first countable S -spaces.

Theorem 20. Assume $2^{\aleph_1} = \aleph_2$. There is a c.c.c. poset Q of size \aleph_2 such that after forcing with Q and then any c.c.c. poset P , there is a first countable perfectly normal hereditarily separable space which is not Lindelöf.

This does it, since one can start e.g., with L and in the Q extension let $P = P_1 * (\dot{P}_2 \times S)$, where P_1 is the forcing for adding a Cohen real, which forces a coherent Souslin tree S [35], and \dot{P}_2 forces $\text{MA}_{\omega_1}(S)$. Since \dot{P}_2 preserves S , $\dot{P}_2 \times S$ and hence P is c.c.c.. In order to force $\text{MA}_{\omega_1}(S)$ we need $2^{\aleph_1} \leq \aleph_2$, but $Q * P_1$ preserves this. \square

To see that Theorem 20 holds we need just to assemble results of others.

Lemma 21 [20]. $\text{MA}_{\omega_1}(S)[S]$ implies $\mathfrak{b} > \aleph_1$.

Lemma 22 [7]. $\mathfrak{b} > \aleph_1$ implies that in a first countable regular space of size \aleph_1 , two disjoint closed sets, one of which is countable, have disjoint open sets around them.

Lemma 23 [29]. $2^{\aleph_1} = \aleph_2$ implies there is a c.c.c. poset Q of size \aleph_2 such that after forcing with Q and then any c.c.c. poset P , there is a first countable 0-dimensional space of size \aleph_1 in which every open set is countable or cocountable.

It just remains to show Soukup's space has the desired properties in our model. He notes it is hereditarily separable but not Lindelöf; by Lemma 22, it is hereditarily normal. Without loss of generality, by passing to a subspace if necessary, we may assume the space is locally countable. But as Roitman [27] notes on p. 314, *a countable subset of a locally countable space is a G_δ* . Since cocountable sets are also G_δ 's we see that the space is perfectly normal.

5 A problem of Nyikos

Next, we deal with a tangentially related problem. In [26], Nyikos raises the question of whether there is a separable, hereditarily normal, locally compact space of cardinality \aleph_1 . He observes that a model in which there are no Q -sets and no locally compact first countable S -spaces would have no such space. In [9], such a model is produced. $\text{PFA}(S)[S]$ also implies these two assertions, so it also implies that there is no such space. To see this, note that a Q -set enables the construction of a locally compact normal space which is not \aleph_1 -collectionwise Hausdorff, while Σ implies there are no (locally) compact S -spaces. \square

6 Some Problems

The referee has asked whether PMEA (the Product Measure Extension Axiom) implies locally compact perfectly normal spaces are paracompact, noting that PMEA implies locally compact normal metalindelöf spaces are paracompact [6]. I do not know the answer to this. However, the reason PMEA implies locally compact normal metalindelöf spaces are paracompact is simply that it implies normal spaces of character $< 2^{\aleph_0}$ are collectionwise normal, whence one gets locally compact normal spaces are \aleph_1 -collectionwise Hausdorff by the usual Watson reduction [37]. As noted earlier, that is enough to make locally compact normal metalindelöf spaces paracompact. What one would need in addition to \aleph_1 -collectionwise Hausdorffness in order to make locally compact perfectly normal spaces paracompact is the non-existence of compact L -spaces plus Σ . It is not known if the former holds under PMEA, although both it and PMEA will hold if one adds strongly compact many random reals over a model of MA_{ω_1} . (The latter is a well-known result of Kunen (see [12]), while the former is in [36].)

The question of whether Σ holds in this model is a stronger version of the unsolved problem of whether there are compact S -spaces in the model obtained by adjoining \aleph_2 random reals to a model of MA_{ω_1} . For several years, this was the preferred approach toward solving Katětov's problem, before $\text{MA}_{\omega_1}(S)[S]$ turned out to be the way to go [24].

The referee also asked whether the Abraham-Todorćevic example of a first countable S -space indestructible under countable chain condition forcing [1] exists under $\text{MA}_{\omega_1}(S)[S]$. Indeed, they start with a model of GCH and do a countable chain condition iteration to construct their example. One can then force with a countable chain condition poset to get $\text{MA}_{\omega_1}(S)[S]$, so this gives another proof that $\text{MA}_{\omega_1}(S)[S]$ does not imply there are no first countable S -spaces. I conjecture however that $\text{PFA}(S)[S]$ implies there are no S -spaces.

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