COMPACT SPACES, ELEMENTARY SUBMODELS, AND THE COUNTABLE CHAIN CONDITION, II

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ABSTRACT. Given a space $\langle X, \mathcal{T} \rangle$ in an elementary submodel of $H(\theta)$, define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. It is established that if X_M is compact and satisfies the countable chain condition, while X is not scattered and has cardinality less than the first inaccessible cardinal, then $X = X_M$. If the character of X_M is a member of M, then "inaccessible" may be replaced by "1-extendible".

This paper continues the line of research of [T_1], [T_2], [JT_2], [Ku], [T_3], [JLT], in which the question of which topological spaces are determined by their compact reflections in elementary submodels is investigated.

Given a space $\langle X, \mathcal{T} \rangle$ in an elementary submodel of $H(\theta)$, we define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. See [$JT_1$] for the basic properties of such X_M's. Since $H(\theta)$ is supposed to be a stand-in for the universe V (see e.g. [JW, chapter 24]), assume θ is regular and "sufficiently large". As a concrete manifestation of largeness, we will consider θ to have cardinality greater than all finite iterations of the power-set function starting with X.

If X_M is compact T_2 (in fact, we shall assume all spaces are T_2), this constrains X to the point that simple additional topological hypotheses on X_M ensure that $X_M = X$ [JT_2]. When powers of the two-point discrete space D are considered, the situation is more complicated: roughly, for κ below very large cardinals, X_M homeomorphic to D^κ implies $X_M = X$, but this is not the case above such large cardinals [T_2], [JT_2], [Ku]. This was generalized to continuous images of powers of D (dyadic compacta) in [T_3] and to compact spaces with regular open algebras isomorphic to those of dyadic compacta in [JLT]. In [JT_2], the question of whether a generalization to compact spaces satisfying the countable chain condition was consistent was raised. In [JLT] it was proved, assuming the Generalized Continuum Hypothesis and the negation of a weak version of Chang's Conjecture, that if X was not scattered, X_M was compact and satisfied the countable chain condition, and the character of X_M was in M, then $X_M = X$. This result was somewhat unsatisfying in that it was not clear whether the

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GCH was needed, or whether this result really needed more "anti-Chang" power than the D^κ results. In this paper we remedy both defects:

Theorem 1. *If X is not scattered and has cardinality less than the first inaccessible, while X_M is compact and satisfies the countable chain condition, then $X = X_M$. If the character of X_M is in M, "inaccessible" can be replaced by "1-extendible".*

Rather weak set-theoretic assumptions (e.g. the non-existence of $0^\#$) imply there are no 1-extendible cardinals. We refer the reader to [Ku] or [K] for the definition of 1-extendibility, since we won't use it. Suffice it to say that a 1-extendible cardinal λ is inaccessible and in fact is the λth measurable cardinal.

The proof will use ingredients similar to those in [JLT], but packaged somewhat differently. The following definition is convenient.

Definition [Ku]. *A compact space X is* **squashable** *if there is an M containing X such that X_M is compact and not equal to X.*

Although X_M can be compact without equalling X – see [JT_2] for several examples – its compactness does entail the compactness of X:

Lemma 2 [Ju]. *If X_M is compact, so is X, and there is a perfect map from X onto X_M.*

Kunen [Ku] improved previous work [T_2], [JT_2] to get the following result:

Lemma 3 [Ku]. *If κ is less than or equal to the first 1-extendible cardinal, then D^κ is not squashable.*

A key concept in Kunen's work is the following:

Definition. *A* **λ-Čech-Pospíšil tree** *in a space X is a tree $\mathcal{K} = \{K_s : s \in {}^{\leq\lambda}2\}$ satisfying:*

1. *Each K_s is non-empty and closed in X.*
2. *$s \subseteq t$ implies $K_s \supseteq K_t$.*
3. *$K_{s^\frown 0} \cap K_{s^\frown 1} = \emptyset$.*
4. *If the length of $s = \gamma$, a limit ordinal, then $K_s = \bigcap_{\alpha<\gamma} K_{s|\alpha}$.*

Definitions of the standard cardinal invariants we shall be using appear in [H] and [J]. In addition, let us make the following definitions:

Definition. *$sc(X)$, the* **scattering number** *of X, is the least cardinal κ such that for each closed subset F of X, there is an $x \in F$ with $\chi(x, F) < \kappa$. $m(X)$, the* **mapping number** *of X, is the least cardinal κ such that for each closed subset F of X, there is an $x \in F$ with $\pi\chi(x, F) < \kappa$.*

The reason for the name of $m(X)$ will be evident from Lemma 6 below. Čech and Pospíšil (see e.g. [J, 3.16]) proved:

Proposition 4. *If X is compact and $\chi(p, X) \geq \lambda$ for all $p \in X$, then there is a λ-Čech-Pospíšil tree in X and hence $|X| \geq 2^\lambda$.*

Kunen translated this into the language of submodels; a reformulation of his work yields:

Lemma 5 [JLT]. *If X_M is compact, $\lambda < sc(X)$ and $\lambda + 1 \subseteq M$, then $2^\lambda \subseteq M$.*

Here is a different array of closed sets, and an important result of Šapirovskiĭ (see e.g. [J, 3.18]):

Definition. *A* κ **-dyadic system** *in a space* X *is a family* $\{\langle F^0_\alpha, F^1_\alpha\rangle : \alpha < \kappa\}$ *of pairs of non-empty closed subsets of* X *such that:*

(a) $F^0_\alpha \cap F^1_\alpha = 0$ *for all* $\alpha < \kappa$,
(b) $F_\varepsilon = \bigcap\{F^{\varepsilon(\alpha)}_\alpha : \alpha \in \mathrm{dom}\ \varepsilon\}$, *for each finite partial function* ε *from* κ *to* 2.

Lemma 6. *The following conditions are equivalent for compact* T_2 *spaces:*

i) X *can be mapped continuously onto* I^κ,
ii) *there is a closed* $F \subseteq X$ *which can be mapped onto* D^κ,
iii) *there is a closed* $F \subseteq X$ *with* $\pi\chi(x, F) \geq \kappa$ *for each* $x \in F$,
iv) *there is a* κ*-dyadic system in* X.

Using Lemma 6, we can vary Lemma 5 to obtain:

Lemma 7 [essentially in JT]. *If* X_M *is compact,* $\lambda \in M$, $\lambda < m(X)$, *and* D^λ *is not squashable, then* $2^\lambda \subseteq M$.

We give the short, instructive proof. Since $\lambda < m(X)$, take $F \in M$, a closed subset of X, and $f \in M$ mapping F onto D^λ. Then f restricted to $F \cap M$ maps F_M onto $(D^\lambda)_M$, so $(D^\lambda)_M$ is compact. But then $(D^\lambda)_M = D^\lambda$ and so $2^\lambda \subseteq M$.

It is sometimes useful to draw conclusions about the existence of nice dense sets from information about $sc(X)$ or $m(X)$. Recall $D \subseteq X$ is $\boldsymbol{G_\delta}$**-dense** if D meets every non-empty G_δ subset of X. The following result comes from [JLT] and [T_3], but is likely folklore.

Lemma 8. *Suppose* X *is compact.*

a) If $sc(X) = \kappa$, then $\{x : \chi(x, X) < \kappa \cdot \aleph_0\}$ is G_δ-dense in X.
b) If $m(X) = \kappa$, then $\{x : \pi\chi(x, X) < \kappa \cdot \aleph_0\}$ is G_δ-dense in X.

Proof. If $sc(X) = \kappa$, take a non-empty G_δ set G with $\chi(x, X) < \kappa$, for each $x \in G$. Then take a non-empty compact G_δ K containing x, with $K \subseteq G$. Then for $x \in K$, we have $\chi(x, X) \leq \chi(x, K) \cdot \chi(K, X)$. Since $\chi(K, X) \leq \aleph_0$ and $\chi(x, K) < \kappa$, it follows that $\chi(x, X) < \kappa \cdot \aleph_0$. Similarly, $\pi\chi(x, X) \leq \pi\chi(x, K) \cdot \chi(K, X)$, so $\pi\chi(x, X) < \kappa \cdot \aleph_0$ is obtained, proving the second assertion.

It is useful to calculate how $sc(X)$ and $m(X)$ relate to some of the other cardinal invariants of X:

Lemma 9.

a) $m(X) \leq sc(X)$;
b) $sc(X) \leq w(X)^+$;
c) *For* X *compact,* $sc(X) \leq |X|^+$;
d) *For* X *regular,* $w(X) \leq \pi\chi(X)^{c(X)}$;
e) *For* X *regular,* $w(X) \leq m(X)^{c(X)}$.

Proof. The first is clear from the definitions. The second is obvious, since if $w(X) = \lambda$, each closed set has a point of character less than λ^+. The third is because $w(X) \leq |X|$ for compact spaces. The fourth is a well-known result of Šapirovskiĭ – see e.g. [H]. For the fifth, by Lemma 8, X has a dense set D of points, each of π-character less than $m(X)$. Then $\pi w(D) = \pi w(X)$, $c(D) = c(X)$, and by d), $\pi w(D) \leq \pi\chi(D)^{c(D)}$. For $x \in D$, $\pi\chi(x, D) = \pi\chi(x, X)$, so $\pi\chi(D) \leq m(X)$. Then $\pi w(D) \leq m(X)^{c(D)}$, so $\pi w(X) \leq m(X)^{c(X)}$, and therefore $w(X) \leq m(X)^{c(X)}$.

We now have most of the ingredients needed to prove Theorem 1. In order to prove $X = X_M$, we will quote a result from [JT$_2$]:

Lemma 10. *Let $X \in M$ be compact. If $\chi(X) \subseteq M$, then $X = X_M$.*

Proof of Theorem 1. There are several cases to consider, depending on what kind of a cardinal $m(X) = \kappa$ is. Note that if $|X| <$ the first 1-extendible, so is $m(X)$.

Case 1. $\kappa \leq 2^{\aleph_0}$.

This will be excluded by Lemma 3 and the following result. Recall a space is **scattered** if each subspace has a point isolated in it.

Lemma 11 [J, 3.17]. *If X is compact and is not scattered, then $m(X) > \aleph_0$.*

A useful corollary of this is:

Lemma 12. *If X_M is compact and satisfies the countable chain condition, and X is not scattered, then X satisfies the countable chain condition.*

Proof of Lemma 12. As observed in [JT$_2$], it suffices to prove that $\omega_1 \subseteq M$, which follows from Lemmas 2, 3, 7 and 11.

Now if $\kappa \leq 2^{\aleph_0}$, $\chi(X) \leq w(X) \leq \kappa^{\aleph_0} \leq 2^{\aleph_0} \subseteq M$, so Case 1 is established.

Case 2. $\kappa > 2^{\aleph_0}$, and there is a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$.

Since X and hence $\kappa \in M$, there is such a $\lambda \in M$. Then by Lemmas 3 and 7, $2^\lambda \subseteq M$. $\kappa^{\aleph_0} \leq 2^\lambda$, so $w(X) \subseteq M$ by Lemma 9 and $X = X_M$ by Lemma 10.

Case 3. κ is a strong limit, $\aleph_0 < cf(\kappa) < \kappa$.

As before, $w(X) \leq \kappa^{\aleph_0} = \kappa$. Let $\lambda = cf(\kappa)$. $\lambda \in M$ and we may take $\{\lambda_\alpha\}_{\alpha<\lambda}$ in M converging to κ. Since $\lambda < \kappa$, by Lemma 7, 2^λ and hence $\lambda \subseteq M$. But then each $\lambda_\alpha \in M$, so $2^{\lambda_\alpha} \subseteq M$. But then $2^{<\kappa}$ and hence $\kappa \subseteq M$ and we are done.

Case 4. $cf(\kappa) = w$, κ a strong limit.

This is the delicate case, and is the only one in which we need to consider character as well as π-character. By the same proof as for Case 3, note we can conclude that $\kappa \subseteq M$. If $sc(X) > \kappa$, then by Lemma 5, $2^\kappa \subseteq M$. Now by Lemma 9, $w(X) \leq \kappa^{\aleph_0} \leq 2^\kappa$, so we are done by Lemma 10. If $sc(X) = \kappa$, we proceed as in [JLT], letting $\{\kappa_n\}_{n<\omega}$ be an increasing sequence in M of uncountable cardinals in M converging to κ. Let $E_n = \{x \in X : \chi(x, X) < \kappa_n\}$. Let $F_n = \overline{E}_n$. Claim $X = \bigcup_{n<\omega} F_n$. For if not, $X - \bigcup_{n<\omega} F_n = \bigcap_{n<\omega} (X - F_n) \neq 0$ and would have to intersect $\bigcup_{n<\omega} E_n$ by Lemma 8. It thus will suffice to prove that $(F_n)_M = F_n$, for every n.

The reason we need to use character rather than π-character here is the following observation from [J, p. 21] used in [JLT]:

Lemma 13. $\{p \in Y : \chi(p, Y) < \lambda\} \leq 2^{c(Y)\cdot\lambda}$.

Thus $|E_n| \leq 2^{\aleph_0\cdot\kappa_n} = 2^{\kappa_n}$. Since $w(Z) \leq 2^{d(Z)}$ for regular Z, $w(F_n) \leq 2^{2^{\kappa_n}} < \kappa$. Since $\kappa \subseteq M$, each $(F_n)_M = F_n$, and hence $X_M = X$.

We can prove the second part of Theorem 1 by closer analysis. Observe that the proof of Case 3 would work for inaccessible $\kappa = m(X)$ if we knew that $|M \cap \kappa| = \kappa$, for then

$2^{<\kappa} = \Sigma\{2^\mu : \mu < \kappa,\ \mu \in M\}$. If $\kappa = m(X)$ is inaccessible, then $\kappa^{\aleph_0} = \kappa$. Let $\lambda = \chi(X_M)$. If $\lambda < \kappa$, then as usual, we conclude that $2^\lambda \subseteq M$. In this case $X = X_M$ by the following result from [T$_3$], generalizing [JT$_2$]:

Lemma 14. *If X_M is compact, $\chi(X_M) \leq \mu$ and $2^\mu \subseteq M$, then $X = X_M$.*

Suppose then that $\kappa \leq \lambda$. Then applying Lemma 9, we have:

$$\kappa \leq \lambda \leq |M \cap w(X)| \leq |M \cap \kappa^{\aleph_0}| = |M \cap \kappa|.$$

Hence $|M \cap \kappa| = \kappa$ and we are done.

Remarks. Theorem 1 is a considerable improvement over [JLT], where GCH and the negation of a form of Chang's Conjecture were needed to draw essentially the same conclusion. Both that paper's results and ours use the annoying hypothesis that $\chi(X_M) \in M$. We do not know whether it is necessary, especially since we avoided it except for inaccessible κ. "Character" is not special; e.g. $d(X_M)$ or $w(X_M)$ would also work.

The bound given by the 1-extendible cardinal can actually be slightly increased – see [Ku]. The requirement in Theorem 1 that X not be scattered cannot be removed: in [JT$_2$] it is proved that every uncountable compact scattered space is squashable. We could however replace it by requiring that X_M not be scattered. To see this, apply Lemmas 2 and 11 to conclude that this would also imply $m(X) > \aleph_0$.

There are a number of ways of packaging the content of the proof of Theorem 1. Here is another one:

Corollary 15. *Suppose there is a non-scattered compact space X which can be squashed to a compact X_M satisfying the countable chain condition and having $\chi(X_M) \in M$. Then for some $\lambda < m(X)$, D^λ is squashable.*

One could actually weaken the countable chain condition in our results to instead assume that each disjoint family of open sets has power less than $2^{\aleph_0}$. One cannot do better because of the following example:

Example. Let X be a one-point compactification of the sum of $(2^{\aleph_0})^+$ copies of the unit interval. Let M be a countably closed elementary submodel of size $2^{\aleph_0}$ containing X. Then X_M has no isolated points and is a one-point compactification of $2^{\aleph_0}$ disjoint copies of the unit interval. Thus X is squashed to a compact X_M with no isolated points but with $c(X_M) = 2^{\aleph_0}$.

As in [JLT], we can partially translate our results into the language of Boolean algebras. For example, we have:

Theorem 16. *Suppose $B \in M$ is a countable chain condition, atomless Boolean algebra such that (Stone space of $B)_M$ is compact, and $|B| <$ the first inaccessible. Then $B \subseteq M$.*

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