PROBLEMS ARISING FROM BALOGH'S "LOCALLY NICE SPACES UNDER MARTIN'S AXIOM"

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Abstract. We discuss recent advances extending the paper mentioned in the title, in particular, the consistency proof for "all locally compact perfectly normal spaces are paracompact". As well, we point out mistakes in the published literature concerning metrizability of hereditarily normal manifolds, and set out an approach toward proving the consistency of such metrizability for those of dimension greater than 1.

1. Introduction

Zoli Balogh contributed so much to set-theoretic topology, solving so many classic problems, that probably each of us has a favorite paper; the one I keep going back to is the one of the title. As will be seen from what I write here, it is definitely inspiring my current work.

In 1983, Zoli published a paper [B_1] which unified Baumgartner's result [BMR] that *MA_{ω_1} implies all Aronszajn trees are special* with Szentmiklóssy's result [Sz] that *MA_{ω_1} implies that there are no compact S-spaces.* He applied his proof in establishing a variety of results on locally compact spaces. In particular, he provided some general conditions on locally compact spaces which implied under MA_{ω_1} that they were paracompact, generalizing Mary Ellen Rudin's result [R] that MA_{ω_1} implies perfectly normal manifolds are metrizable. In this note we shall discuss new methods that lead to significant improvements of Balogh's results, and also call attention to some important gaps in the published literature concerning the metrizability of hereditarily normal manifolds.

There are two problems that are particularly noteworthy in sequels to Balogh's paper. One is the question of whether it is consistent that *all locally compact perfectly normal spaces are paracompact*; the other is whether it is consistent that *all hereditarily normal manifolds of dimension greater than* 1 *are metrizable.* The first question should have received more attention than it has, considering e.g. the

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amount of attention the question of whether locally compact normal metacompact spaces are paracompact [W_1], [GK_1], [GK_2] received after Arhangel'skiĭ [A] proved that locally compact perfectly normal metacompact spaces are paracompact. Perhaps it was too bold a conjecture to imagine, given the plethora of consistent counterexamples and the total lack of any conceivable path to a proof. Be that as it may, the problem was raised for the first time in [W_1], again in [W_2], and again in [W_3], where Watson dubbed it his favorite problem and called the prospect of a consistency proof "almost impossible". In several conferences in 2002, Paul Larson and I announced such a consistency proof – at least modulo a supercompact cardinal – but noted our work depended on results announced by S. Todorčević [To] which have not yet been written up. Since we still have not seen all of Todorčević's proof − although his results are plausible, given the methods he uses – we have refrained from submitting our preprint [LT_1] for publication as yet. Since there will be such a long delay from our announcement until its publication, we thought it useful to sketch our part of the proof in this memorial issue. The other problem has an equally long history, dating back to Nyikos' [N_1] and inspiring several of his papers [N_3], [N_4], [N_5] since. In our approach to these problems, the key question has been the attempt to obtain the consequence of $V = L$ that we'll call **CW**: *normal first countable spaces are collectionwise Hausdorff* [F_1], together with Balogh's consequences of MA_{ω_1}. Such a combination solves the first problem; a plausible path to the second requires two other combinatorial propositions as well. As we did in conferences in 2002, we shall briefly sketch our contribution to the perfectly normal proof. We shall then discuss our proposed approach to the solution of the second problem. Finally, we shall introduce a new improvement of the perfectly normal results. With the exception of Theorems 13, 16, and 17, the proofs of results of Larson and/or myself mentioned or sketched below can be found in detail in the preprints [LT_1], [LT_2], and [L_2].

There are several references in this paper to proofs that have not appeared and indeed I have not seen. This is dangerous; the story of Nyikos' theorem referred to below is a cautionary tale. I would have preferred to submit this article a year later, but since it is so fitting for this memorial issue, I decided not to delay it. The reader is encouraged to ingest this material with as many grains of salt as seem appropriate.

2. Locally compact perfectly normal spaces

Balogh proved that MA_{ω_1} implies Σ : *locally countable subspaces of size* $\aleph_1$ *of compact countably tight spaces are* σ*-discrete.* (We will assume all spaces are Hausdorff.) The theorems of Baumgartner and Szentmiklóssy mentioned in the introduction follow easily. Balogh also noted that:

Lemma 1. *The one-point compactification* X^* *of a locally compact countably tight space* X *is countably tight if and only if* X *does not include a perfect pre-image of* ω_1*.*

Since perfect pre-images of ω_1 are not perfectly normal, it follows that the one-point compactification of a locally compact perfectly normal space is countably tight.

Szentmiklóssy also proved a result "dual" to the one of his quoted above: MA_{ω_1} *implies* **L:** *there are no first countable L-spaces, i.e. every first countable regular hereditarily Lindelöf space is hereditarily separable.* Let us now sketch a proof that:

Theorem 2. $\Sigma + L + CW$ *imply every locally compact perfectly normal space is paracompact.*

Following Nyikos [N_2], we have:

Definition. *A space M is of* **Type I** *if $M = \bigcup_{\alpha<\omega_1} M_\alpha$ where the M_α's are open, $\overline{M}_\beta \subseteq M_\alpha$ whenever $\beta < \alpha$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for limit α, and each $\overline{M}_\alpha$ is Lindelöf.*

$B_\alpha = \overline{M}_\alpha - M_\alpha$ is called a **bone**; the **skeleton** of M is the collection of bones; a selection of one point from each bone we call a **bone-scan**.

The relevance of Type I spaces to the problem at hand is the following lemma, which we prove by methods of Balogh [B_1].

Lemma 3. *If X is hereditarily collectionwise Hausdorff, locally hereditarily Lindelöf, and subspaces of X are hereditarily Lindelöf if and only if they are hereditarily separable, then X is the disjoint union of clopen Type I spaces.*

Assuming $\Sigma + L + CW$, it is then not difficult to conclude that a locally compact perfectly normal space is the disjoint union of clopen Type I spaces. Thus, in proving locally compact perfectly normal spaces are paracompact, we may confine ourselves to Type I spaces.

It is easy to establish [N_2]:

Lemma 4. *If the skeleton of a Type I space has a closed unbounded set of empty bones, the space is paracompact.*

and

Lemma 5. *If X is locally hereditarily Lindelöf Type I, then X is hereditarily collectionwise Hausdorff if and only if every discrete subspace misses the elements of a skeleton closed unboundedly often.*

We can now prove Theorem 2. Bone-scans are locally countable in a Type I space and hence – if the space is locally compact – in its one-point compactification. If that space is perfectly normal, its compactification is countably tight, so Σ applies to get the bone-scan σ-discrete. Lemmas 4 and 5 complete the proof.

It is not obvious how to obtain the simultaneous consistency of Σ, L and CW. Σ and L are strong "Souslin-type" consequences of MA_{ω_1} in the sense of [KT], while

CW is contradicted by the "combinatorial" consequence of MA_{ω_1} in the sense of [KT] (i.e. following from MA_{ω_1} (σ-centred)) that there exists a Q-set. However the proof by Larson and Todorčević of the consistency of L with the non-existence of Q-sets [LTo] pointed us in the right direction. Indeed Todorčević [To] announced that there is a model for $\Sigma + L$, which we can modify so that it also satisfies CW.

The model of [LTo] is obtained by starting with a particular kind of Souslin tree – a "coherent" one, obtainable from $\diamondsuit$ for example. One then forces as much as possible of MA_{ω_1} without destroying the coherent tree S, and then forces with S. The model Todorčević uses is obtained via an analogous program, except replacing MA_{ω_1} by PFA. More formally, it is obtained by forcing with S over a model of:

PFA(S): *If P is a proper partial order which doesn't force an uncountable antichain in S, and if $\{D_\xi\}_{\xi<\omega_1}$ is a sequence of dense open subsets of P, then there is a filter $G \subseteq P$ such that $G \cap D_\xi \neq 0$ for each $\xi < \omega_1$.*

The difficult task is to show that PFA(S) is enough to ensure that locally countable subspaces of size $\aleph_1$ of compact spaces with countable tightness constructed by the S-forcing are σ-discrete.

We modify the construction of the model by doing some preliminary forcing before creating the coherent Souslin tree S, forcing PFA(S), and then forcing with S. Given a supercompact κ, we force GCH below κ, follow Laver [La] to make the supercompactness of κ indestructible under κ-directed-closed forcing, and then Easton-force to add λ^+ Cohen subsets of λ for every regular cardinal $\lambda \geq \kappa$. This will establish Fleissner's "$\diamondsuit$ for stationary systems" [F_1] at every regular $\lambda \geq \kappa$. We then proceed as did Todorčević. Since that forcing has the κ-chain condition, $\diamondsuit$ for stationary systems will still hold at every regular $\lambda \geq \kappa = \aleph_2$ [T_1]. It follows by [F_1] that normal first countable $\aleph_1$-collectionwise Hausdorff spaces will be collectionwise Hausdorff. The final observation is:

Lemma 6. *Forcing with a Souslin tree yields a model in which normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

The proof is a bit tricky, but, as usual, the idea is to show that "normalizing" a generic partition yields a separation.

There are numerous other consistency results concerning locally compact normal spaces in [LT_1]; a very striking one is:

Theorem 7. *Σ plus L plus CW imply that locally compact spaces with hereditarily normal squares are metrizable.*

This is proved along the same lines as Theorem 2, using in addition Katětov's Theorem [K] and the solution to his problem obtained in [LTo].

3. Manifolds

The basic reference for the theory of non-metrizable manifolds is Nyikos' [N_2]. Approaching his problem, in [N_3], [N_4], [N_5] he gives a variety of "proofs" that:

If it's consistent that there is a supercompact cardinal, then it's consistent that hereditarily normal hereditarily collectionwise Hausdorff manifolds of dimension greater than 1 *are metrizable.*

The proofs given all depend on the assertion – attributed mistakenly to Shelah – that PFA^+ and the existence of a stationary $S \subseteq \omega_1$ such that $NS_{\omega_1}|S$ is $(\aleph_2, \aleph_2, \aleph_0)$-saturated, (defined below) are compatible. Larson [L$_2$] has shown that they are not. However, Nyikos has salvaged most of his work and applied it to indeed prove the result displayed above. We believe his work can be combined with the ideas of the first part of this note so as to drop the hereditary collectionwise Hausdorffness. We shall discuss such an approach below. First, some definitions, so that we can say exactly what Larson proved.

Definition. *A* **σ-ideal** $\mathcal{J}$ *on* ω_1 *is a collection of subsets of* ω_1 *containing all singletons and closed under countable unions.*

The non-stationary ideal NS_{ω_1} on ω_1 is a σ-ideal, as is $NS_{\omega_1}|S = \{I \subseteq \omega_1 : I \cap S \text{ is non-stationary}\}$, for any stationary $S \subseteq \omega_1$.

Definition. *A σ-ideal $\mathcal{J}$ on ω_1 is* **$(\aleph_2,\aleph_2,\aleph_0)$-saturated** *if whenever* $\{S_\alpha\}_{\alpha<\omega_2} \subseteq \mathcal{J}^+ = \mathcal{P}(\omega_1) - \mathcal{J}$*, there is an* $A \in [\omega_2]^{\omega_2}$ *such that for every* $Z \in [A]^\omega$, $\bigcap_{\alpha \in Z} S_\alpha \in \mathcal{J}^+$.

The consistency of a supercompact suffices for the consistency of $NS_{\omega_1}|S$ being $(\aleph_2, \aleph_2, \aleph_0)$-saturated for some stationary S [Sh, chapter XIII]. It also suffices for the consistency of PFA^+ [Ba]. However, contrary to the claims in [N$_3$], [N$_4$], [N$_5$], such saturation and PFA^+ cannot be obtained simultaneously, since Larson [L$_2$] has proved:

Theorem 8. MA_{ω_1} *plus* $2^{\aleph_0} = \aleph_2$ *implies no σ-ideal on ω_1 is* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated.*

Amazingly, however, within a few weeks of being informed of Larson's result, Nyikos managed to completely eliminate saturation from his proofs and indeed obtained:

Theorem 9 [N$_6$]. *PFA implies every hereditarily normal, hereditarily collectionwise Hausdorff manifold of dimension greater than* 1 *is metrizable.*

Some of the other results in his papers, which attempted to replace hereditary normality plus hereditary collectionwise Hausdorffness by hereditary *strong* collectionwise Hausdorffness are still in question.

Most of the key ideas in Nyikos' new proof are already contained in [N$_3$] and [N$_4$]. Assuming non-metrizability and hence non-paracompactness, the eventual contradiction aimed for is to obtain within the manifold a perfect pre-image of ω_1 having $\aleph_1$ disjoint copies of ω_1. In fact Nyikos proves in [N$_4$] that one can't even get infinitely many such copies. A relevant version of his result is:

Lemma 10 [N$_4$]. *Let X be a hereditarily normal perfect pre-image of ω_1. Then X cannot include an infinite disjoint family of closed, countably compact, non-compact subspaces.*

Note that a countably compact subspace of a manifold is closed, by first countability.

The assumption that the manifold is of dimension greater than one is needed only for the following result:

Lemma 11 [N$_3$]. *A Type I manifold M of dimension greater than 1 has a Type I representation $M = \bigcup_{\alpha<\omega_1} M_\alpha$ such that each point p of any $B_\alpha = \overline{M}_\alpha - M_\alpha$ is contained in a non-trivial continuum $K_p \subseteq B_\alpha$.*

The advantage of this is that, after one subtracts a copy W of ω_1 from M, there are plenty of points in any K_p that are not in W. Copies of ω_1 will be found abundantly in M by using the following consequence [B$_2$] of PFA:

PPI: *Every first countable perfect pre-image of ω_1 includes a copy of ω_1.*

To get the particular pre-images of ω_1 needed, Nyikos uses the axiom $\mathbf{CC}_{22}$, which in [N$_3$] and [EN] is proved to be a consequence of PFA$^+$. However, using results from [H], Nyikos [N$_4$] notes it can now be obtained just from PFA.

Definition. *$\mathcal{I} \subseteq \mathcal{P}(X)$ is an* **ideal** *if every subset of a member of $\mathcal{I}$ is in $\mathcal{I}$, and $\mathcal{I}$ is closed under finite unions. An ideal $\mathcal{I}$ of countable subsets of X is* **countable-covering** *if for each countable $Q \subseteq X$, there are $\{I_n^Q : n \in \omega\}$, each $I_n^Q \in \mathcal{I}$, such that whenever $I \subseteq Q$ and $I \in \mathcal{I}$, then $I \subseteq I_n^Q$ for some n.*

$\mathbf{CC}_{22}$**:** *for each countable-covering ideal $\mathcal{I}$ on a stationary subset of ω_1, either*

(i) there is a stationary $A \subseteq S$ such that $[A]^\omega \subseteq \mathcal{I}$,

or (ii) there is a stationary $B \subseteq S$ such that $B \cap I$ is finite for all $I \in \mathcal{I}$.

What CC$_{22}$ does for us is yield the following:

Lemma 12. *Assume CC$_{22}$. Let $Q = \{q_\alpha : \alpha \in S\}$ be a subset of a bone-scan of a hereditarily collectionwise Hausdorff Type I manifold, S a stationary subset of ω_1. Then there is a stationary $T \subseteq S$ such that $\overline{\{q_\alpha : \alpha \in T\}}$ is a perfect pre-image of ω_1.*

Nyikos shows this by applying CC$_{22}$ to the ideal of countable subsets of Q with compact closure so as to obtain a stationary $T \subseteq S$ such that for every countable $U \subseteq T$, $\overline{\{q_\alpha : \alpha \in U\}}$ is compact. The other alternative provided by CC$_{22}$ yields a stationary $T \subseteq S$ such that $\{q_\alpha : \alpha \in T\}$ is discrete, which is ruled out by the topological hypotheses. He then shows that $\overline{\{q_\alpha : \alpha \in T\}}$ is as desired.

The main line of the proof proceeds by using PFA to get that the manifold is Type I and includes a perfect pre-image of ω_1 and hence a copy W of ω_1. Because

ω_1 is not paracompact, W hits stationarily many bones and so by Lemma 11 we can find continua about a point of W in each of those bones, included in that bone. Taking a point in each of those continua and applying CC_{22}, one gets a perfect pre-image P of ω_1 included in M − W. Urysohn's Lemma then yields a continuous $f : M \to [0, 1]$ sending W to 0 and P to 1. It follows that each of the chosen continua maps onto all of [0, 1]. This allows Nyikos to carefully construct another perfect pre-image of ω_1 — this is the new element of the proof — that meets uncountably many $f^{-1}(y)$'s, such that each of those intersections by CC_{22} and PPI includes a copy of ω_1.

Our plan is to follow Nyikos' proof, but to get hereditarily collectionwise Hausdorffness "for free" from the first countability of manifolds and the assumed hereditary normality. Since it is not his immediate concern, Nyikos does not pay close attention to the amount of MA_{ω_1} he needs, in addition to the two consequences PPI and CC_{22} of PFA. We do pay close attention, since we want to get CW so must avoid full MA_{ω_1}. In fact, in addition to PPI and CC_{22}, all that's really needed besides CW is our by now familiar Σ!

Theorem 13. *Assume Σ, CW, PPI and CC_{22}. Then hereditarily normal manifolds of dimension greater than 1 are metrizable.*

We won't give the proof here since it would require going through Nyikos' proof in detail, but just as an example, let us prove:

Theorem 14. Σ *implies hereditarily collectionwise Hausdorff manifolds are Type I.*

Corollary 15. Σ *plus CW implies hereditarily normal manifolds are Type I.*

Proof. The Corollary is clear, for let M be a hereditarily normal manifold. Then M is first countable, so hereditarily collectionwise Hausdorff.

To prove the theorem, note that manifolds are locally compact and locally metrizable, so locally hereditarily Lindelöf. Since manifolds are connected, it would suffice to show that hereditarily collectionwise Hausdorff ones are disjoint unions of clopen Type I spaces. Note that Lindelöf subspaces of manifolds are metrizable and hence hereditarily separable. We claim that hereditarily separable subspaces are (hereditarily) Lindelöf. For let Y, a subset of a manifold M, be hereditarily separable. Then $\overline{Y}$ has no uncountable discrete subspace, else we could apply hereditary collectionwise Hausdorffness and then trace uncountably many disjoint open sets onto a countable dense set. $\overline{Y}$ is locally compact; its one-point compactification $\overline{Y}^*$ also has no uncountable discrete subspace and so has countable tightness. But if Y were not hereditarily Lindelöf, there would be a locally countable subspace Z of size $\aleph_1$, in Y and hence in $\overline{Y}^*$. But then by Σ, Z would be σ-discrete, and hereditary collectionwise Hausdorffness would yield a contradiction. The proof of the theorem is completed by referring to Lemma 3 above.

Thus our approach to solving Nyikos' problem is to use the model of [LT_1], or possibly a stronger one using MM or $\mathbb{S}_{\max}$ [L_1] instead of PFA, and prove that Σ, CWN, PPI, and CC_{22} hold in this model. These last two tasks remain to be accomplished. However they are reasonable: each can be obtained by proper forcing that doesn't add reals, over a model of CH [E], [EN]. Our task is much easier, now that we don't have to worry about the saturation axioms of [N_3], [N_4], and [N_5]. However we are far from an understanding – e.g. to be able to formulate as an axiom - what are the consequences that hold in these models. The several topologically noteworthy proofs so far using these models have proceeded in several different fashions. In the [LTo] solution to Katětov's problem, Larson and Todorčević show that a weak version of MA_{ω_1} is sufficient to imply the desired topological consequences: there are no first countable L-spaces and no compact first countable S-spaces. Defining $MA_{\omega_1}(S)$ analogously to PFA(S) above, they show that this weak version is obtained by forcing with S over $MA_{\omega_1}(S)$. They also show that forcing with a Souslin tree yields a model in which there are no Q-sets. The proof of Lemma 6 above in [LT_1] is an extension of this second approach. The proof of Σ in [To] on the other hand uses proper forcing with elementary submodels as side conditions to show that Σ holds for spaces with S-names over PFA(S).

Even for countably compact manifolds, Nyikos' problem has not been solved. However further analysis of Nyikos' proof enables one to prove:

Theorem 16. *Assume Σ, CW, and PPI. Then countably compact, hereditarily normal manifolds of dimension greater than* 1 *are metrizable.*

4. Strengthening Σ

Despite the usefulness of Σ, I have found that it plus CW does not appear sufficient to prove some desirable consequences; rather one wants a strengthening of Σ that follows straightforwardly from MA_{ω_1}. One needs then to check that Todorčević's proof works for this strengthening as well.

Here is the requisite strengthening of Σ:

$\mathbf{\Sigma^+}$: *Suppose X is a countably tight compact space, $\mathcal{L} = \{L_\alpha\}_{\alpha<\omega_1}$ a collection of disjoint compact sets such that each L_α has a neighborhood that meets only countably many L_β's, and $\mathcal{V}$ is a family of $\leq \aleph_1$ open subsets of X such that:*

a) $\bigcup \mathcal{L} \subseteq \bigcup \mathcal{V}$,

b) For every $V \in \mathcal{V}$ there is an open U_V such that $\overline{V} \subseteq U_V$ and U_V meets only countably many members of $\mathcal{L}$.

Then $\mathcal{L} = \bigcup_{n<\omega} \mathcal{L}_n$, *where each $\mathcal{L}_n$ is a discrete collection in* $\bigcup \mathcal{V}$.

Σ^+ can be used to show that:

Theorem 17 [T_2]. In the model of [LT_1], locally compact normal spaces which do not include a perfect pre-image of ω_1 are collectionwise Hausdorff.

The obvious attempt to replicate the ideas in [W_1] in the context of [LT_1] breaks down because it is not known e.g. whether (locally compact) normal spaces of character $\leq \aleph_1$ are $\aleph_1$-collectionwise Hausdorff after forcing with a Souslin tree. Nonetheless, a proof of Theorem 17 can be accomplished. Using Watson's character reduction method, one can expand the elements of a closed discrete set of power $\aleph_1$ in a locally compact normal space to a locally countable collection of compact sets of countable character. Using Σ^+, this collection can be expressed as the union of countably many discrete collections.

We can then use the same combinatorics that came from the Souslin tree forcing that showed normal first countable spaces are $\aleph_1$-collectionwise Hausdorff in order to separate any of these discrete collections. Normality and standard techniques then enable us to separate the original closed discrete subspace.

To use Σ or Σ^+, we need to know X^* is countably tight. We have:

Lemma 18 [$\mathbf{LT}_2$]. *If X is locally compact and does not include a perfect pre-image of ω_1, then X^* is countably tight.*

As mentioned previously, this was proved for countably tight X in [B_1]. It was asserted without proof in [B_3].

Todorčević's use of a supercompact in obtaining Σ (and presumably Σ^+) can probably be dispensed with. It is likely that, at worst, an inaccessible is needed. However, using a supercompact, stronger results can be obtained. Fleissner [F_2] introduced a stationary-set-reflection axiom, Axiom R, which he obtained from a supercompact and which Balogh [B_3] used to prove:

Lemma 19. *Assume MA_{ω_1} + Axiom R. Then if X is locally compact, hereditarily strongly $\aleph_1$-collectionwise Hausdorff, and does not include a perfect pre-image of ω_1, then X is paracompact.*

Axiom R holds in a strengthening [LT_2] of the model of [LT_1]. An analysis of the proof of Lemma 19 shows that "Σ^+" would suffice, instead of "MA_{ω_1}". Assuming the proof of Todorčević which we have not yet seen, we can then get the extraordinary:

Theorem 20. *If it is consistent there is a supercompact cardinal, it's consistent that locally compact, hereditarily normal spaces that do not include a perfect pre-image of ω_1 are hereditarily paracompact.*

The proof appears in the preprint [LT_2], the submission of which again waits on the materialization of [To].

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