Perfectly Normal Non-metrizable Non-Archimedean Spaces are Generalized Souslin Lines

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Abstract

In this paper we prove the equivalence between the existence of perfectly normal, non-metrizable, non-archimedean spaces and the existence of "generalized Souslin lines", *i.e.,* linearly ordered spaces in which every collection of disjoint open intervals is σ-discrete, but which do not have a σ-discrete dense set. The key ingredient is the observation that every first countable linearly ordered space has a dense non-archimedean subspace.

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These results are part of the Ph.D. thesis of the first author [Q_1] produced under the supervision of the second author. Publication was delayed for a variety of reasons, but LOTS (of) experts have asked that they appear, even at this late date.

Essentially the same question about linearly ordered spaces has been asked by several authors; the following theorem establishes that equivalence and adds a new version. Necessary definitions are presented immediately thereafter.

Theorem 1 *The following are equivalent:*

a) *there is a perfectly normal non-archimedean non-metrizable space,*

b) *there is a perfect linearly ordered space which does not have a σ-discrete dense subspace,*

c) *there is a linearly ordered space in which every disjoint collection of convex open sets is σ-discrete, but which does not have a σ-discrete dense subspace,*

d) *there is a linearly ordered space without isolated points which does not have a σ-discrete dense subspace, but every nowhere dense subspace of it does have such a subspace.*

Definition. *A linearly ordered topological space* or a *LOTS* is a space whose topology can be induced by a linear order, taking the family of open rays as a subbase.

We shall denote an arbitrary LOTS by L.

Definition. A topological space is a *generalized ordered space* or a *GO* space if it can be topologically embedded into a LOTS.

Definition. *A non-archimedean space,* (X, T), is a space with a topological base T which is a tree under the inclusion order.

Definition. Let T be a tree. The *branch space of* T *is* $(X(T), T)$, *where* $X(T) = \{b \subseteq T : b \text{ is a branch of } T\}$ and the basic open sets are determined by the nodes of b, *i.e.,* if $t \in b$, then $\{b' \in X(T) : t \in b'\}$ is a basic open set about b.

Definition. A family of sets of a space X is called *σ-discrete* if it can be partitioned into countably many discrete subfamilies.

Definition. A subset C of a LOTS is *convex* if $(x, y) \subseteq C$ for any $x, y \in C$.

Definition. L is called *σ-discrete cc (or σ-dcc)* if its antichains of open sets are all σ-discrete.

Definition. L is called *convex σ-discrete cc (or convex σ-dcc)* if its antichains of convex open sets are all σ-discrete.
Definition. A subspace is called *σ-discrete* if it can be partitioned into countably many closed discrete subspaces.
Definition. A subspace is called *σ-relatively-discrete* if it can be partitioned into countably many discrete subspaces.
Definition. X is called a K_0 space [T] if it has a σ-relatively-discrete dense subspace.
Definition. X is called a K_0' space if it has a σ-discrete dense subspace.

Clearly, σ-dcc is a generalization of ccc, and both K_0 and K_0' are generalizations of separable. Surprisingly, there is an example of a LOTS which is K_0' (in fact metrizable) but has an antichain of open sets which is not σ-discrete [F]; however we shall see that if a LOTS is K_0', it is convex σ-dcc. The following question is due to S. Watson [W, Problem 107], [BDDGN-RTW]:
Question. Is the existence of a LOTS which is σ-dcc but not K_0' equivalent to the existence of a Souslin tree?

We shall prove the answer is "yes". In view of the example of [F] mentioned above, the following variation of Watson's question (which, he informs us, is what he really meant) is natural:
Question. Is there a LOTS which is convex σ-dcc but not K_0'?

We call such a LOTS a *generalized Souslin line.*

We call a perfectly normal non-metrizable non-archimedean space an *archvillain.* Nyikos [N] asked whether archvillains exist. In [T], [Q] and [QT] the existence of archvillains is discussed. The consistency of the existence of archvillains of size $\aleph_2$, but none of size $\aleph_1$, is proved in [Q], improving earlier results of [T].

The question of whether there is a perfect LOTS which is not K_0 is due to Maurice and van Wouwe (mentioned in BL). The fourth equivalent in Theorem 1 involves a natural but new concept.
Definition. A LOTS without isolated points which is not K_0' but in which every nowhere dense subspace is K_0' is called a *generalized Lusin line.*

One can also ask whether any of the four equivalent kinds of spaces in Theorem 1 exists with a point-countable base. Nyikos [N] asked this for (a) and noted that the Souslin line obtained from a Souslin tree in the usual way is one. Heath (mentioned in BL) asked if there is a perfectly normal non-metrizable LOTS with a point-countable base. This is also equivalent to the point-countable base version of the question in Theorem 1.

Theorem 2 *The following are equivalent:*

a) *there is an archvillain with a point-countable base,*

b) *there is a perfect LOTS with a point-countable base which is not* K_0.

c) *there is a generalized Souslin line with a point-countable base,*

d) *there is a generalized Lusin line with a point-countable base,*

e) *there is a perfectly normal non-metrizable LOTS with a point-countable base.*

It is not known whether it is consistent that there are no archvillains or even none with point-countable bases. It is also not known whether the existence of an archvillain implies the existence of one with a point-countable base. However, using results from [QT], we shall prove

Theorem 3 *If it is consistent that there is a supercompact cardinal, then it is consistent that every archvillain includes one with a point-countable base.*

1 The existence of generalized Souslin lines.

In this section we prove our main result – the equivalence of (a) and (c) of Theorem 1. Our task is made easier by the following result of Faber, which incidentally shows (b) and (c) of Theorem 1 are equivalent.

Lemma 4 *[F, p.38]. Let X be a GO space. Then the following are equivalent:*

a) *X is perfectly normal,*

b) *X is convex σ-dcc,*

c) *every relatively discrete subspace of X is σ-discrete in X.*

Note that c) implies K_0 and K_0' are equivalent for perfect GO spaces. (Recall all LOTS are hereditarily normal.)

It is now easy to prove

Proposition 5 *([P] in essence). An archvillain is a generalized Souslin line.*

Proof. An archvillain is a perfectly normal non-archimedean space, hence a LOTS [P]. It is not K_0 (and hence not K_0') because K_0 perfectly normal non-archimedean spaces are metrizable [T]. However it is convex $\sigma - dcc$ by Lemma 4.

Proposition 6 *Every generalized Souslin line includes an archvillain.*

This will follow from the following useful result.

Theorem 7 *Every first countable LOTS includes a dense non-archimedean subspace.*

Proof. We perform a construction like the construction of a Souslin tree from a Souslin line. We construct a tree of non-empty open intervals of L by recursion, so that the branch space of T is dense in L. L may have isolated points, but these are also open intervals, except for initial or final isolated points, which can conventionally be written as (∞, a) or (b, ∞), so we will also consider them to be open intervals. We form the 0*'th* level T_0 of the tree by taking the set of isolated points of L and expanding it to a maximal disjoint collection of non-empty open intervals of L. Then every $t \in T_0$ which is not isolated in fact contains no isolated points. For each such non-isolated t, pick any point $x(t)$ of t which is not adjacent to some other point of L.

For each such $x(t)$, pick a strictly decreasing sequence of open intervals $\{(a_n(t), b_n(t)) : n < \omega\}$ included in t and converging to $x(t)$, *i.e.,* $a(t) < \ldots < a_n(t) < a_{n+1}(t) < x(t) < b_{n+1}(t) < b_n(t) < \ldots < b(t)$ and $x(t) = sup\{a_n(t) : n < \omega\} = inf\{a_n(t) : n < \omega\}$. We can do this because L is first countable and t has no isolated points. We call $x(t)$ a *centrepoint* of t.

To construct level T_1, if t was isolated, its successor is itself. If t contained no isolated points, we take the immediate successors of t to be $(a(t), a_0(t))$, $(a_0(t), b_0(t))$, $(b_0(t), b(t))$. Then $\bigcup T_1$ is dense in $\bigcup T_0$. To construct level T_n, $n > 1$, each isolated point succeeds itself as do all elements except those of form $(a_{n-1}(t)), b_{n-1}(t))$. These are succeeded by $(a_{n-1}(t), a_n(t))$, $(a_n(t), b_n(t))$, $(b_n(t), b_{n-1}(t))$.

At limit α, by construction, each branch will either have intersection empty, a single point, or an interval (which may be an isolated point). The interiors of those intervals will be the elements of level α. We choose centre points for the non-isolated open intervals as at level 0, and then continue the tree as we did for the finite levels. We continue the construction until we get to a limit level where all branches have empty or singleton intersection.

Note that each element of the tree is either isolated or contains a centrepoint, since it acquires one at the next limit level, if it did not have one already.

Let X be the set of isolated points of L together with the set of centrepoints. Since the basic open sets of the branch space (X,T) are of form $X \cap t$, where $t \in T$ is an open interval of L, (X,T) is dense in L. Suppose $I = (a,b)$ is an open interval of L; we shall prove that I includes some element of T, for if not, we will find a non-isolated open interval included in the intersection of a cofinal branch of T, violating our construction. If I contains any isolated points, then it includes an element of T_0, so assume it has no isolated points.

At level 0, I will meet some element of T_0 since T_0 is a maximal antichain. Let $t = (x,y) \in T_0$ such that $I \cap t \neq \emptyset$. Let $I' = (a,b) \cap (x,y)$. Supposing I does not include t, then either $I = I'$ or $I' = (x,b)$ or $I' = (a,y)$. Suppose $I' = (a,y)$, *i.e.,* $x < a < y < b$. The other cases are similar. We proceed with I'. Let γ be the height of T. For $\alpha < \gamma$, assume that I' is included in some element of T_β, for each $\beta < \alpha$. We claim that I' is included in some element of T_α as well. But then $\{t \in T : \ t \supset I'\}$ is a branch of length γ with intersection neither empty nor singleton, a contradiction.

To prove the claim, consider first the case $\alpha = \beta + 1$. So there is a $t_\beta = (x_\beta, y_\beta) \in T_\beta$ with $I' \subset t_\beta$. If $t_\beta \in T_\alpha$, we are done; otherwise t_β was cut into $(x_\beta, a_n), (a_n, b_n), (b_n, y_\beta)$ at level α. Since $I' \subset t_\beta$, I' must meet at least one of these. We are supposing I' does not include any of them. If I' is not included in any of them either, we will reach a contradiction. Suppose, for example, that $I' \cap (b_n, y_\beta) \neq 0$. If $b_n \in I'$, then $a < b_n < y \leq y_\beta$. So $y < y_\beta$, else $I' \supset (b_n, y_\beta)$. Then $x < b_n < y < y_\beta$. But this contradicts that (x,y) and (b_n, y_β) are either disjoint or one includes the other, since T is a tree. If on the other hand, $y_\beta \in I'$, then $b_n < a < y_\beta < y$. But $I' \subset t_\beta$, so $y \leq y_\beta$, contradiction.

The case when $I' \cap (x_\beta, a_n) \neq 0$, is similar to the first case, so suppose $I' \cap (a_n, b_n) \neq 0$. If $a_n < a < b_n < y_\beta$, then $y < y_\beta$ else $(b_n, y_\beta) \subset (a,y)$. Similarly, $y > b_n$ else $(a_n, b_n) \supset I'$. So $a_n < a < b_n < y < y_\beta$. But $x < a$, so by treeness, $x \leq a_n$. Again by treeness, $x_\beta \leq x$. Thus $x_\beta \leq x \leq a_n < a < b_n < y < y_\beta$. But (x_β, y_β) is on a later level of the tree than (x,y), so if $(x_\beta, y_\beta) \supset (x,y), (x_\beta, y_\beta) = (x,y)$, contradiction.

The other cases are similar.

If α is a limit ordinal such that for each $\beta < \alpha$, $t_\beta = (x_\beta, y_\beta) \in T_\beta$ such that $t_\beta \supset I'$, then $I' \subset \bigcap\{t_\beta : \beta < \alpha\}$. So the intersection of the chain $\{t_\beta : \ \beta < \alpha\}$ is a non-empty non-singleton convex set. Hence, we have put an antichain of open intervals at the αth level which is maximal in the intersection. So there is a $t' = (x', y') \in T_\alpha$ such that $t' \cap I' \neq \emptyset$. As in the successor case, we conclude some member of T_α includes I'.

Thus, every open interval includes some element of T and therefore contains at least one centrepoint. Therefore X is dense in L.

We can now easily prove Proposition 6.

Proof. Given a generalized Souslin line L, let X be a dense non-archimedean subspace. L is perfectly normal, so X is. We claim X is not metrizable. If it were, it would be K_0. Then L would be K_0. By Lemma 4 it would then be K_0', contradiction.

Note. A previous version of this paper claimed Theorem 7 without the restriction of first countability. We are grateful to A. Jones for providing a counterexample [J].

2 σ-dcc versus convex σ-dc

Since separability implies the countable chain condition, one might expect K_0' to imply σ-dcc. It doesn't but we do have

Lemma 8 *[BL]. K_0' GO spaces are perfectly normal, hence convex σ-dcc.*

Lemma 8 yields a characterization of metrizability in non-archimedean spaces:

Theorem 9 *A non-archimedean space is metrizable iff it is K_0'.*

Proof. A perfectly normal non-archimedean space is metrizable iff it is K_0' [T]. As noted earlier, K_0 is equivalent to K_0' for perfect GO spaces. ■

Let us also mention the characterization in [BL] of metrizability in GO spaces:

Proposition 10 *A GO space is metrizable iff it is K_0' and has a point-countable base.*

The Sorgenfrey line shows that the latter condition cannot be dropped.

As we said above, K_0' does not imply σ-dcc, and hence σ-dcc and convex σ-dcc are not equivalent. There is an example in [F]; we give a non-archimedean one. Note every metrizable space is K_0' and perfectly normal.

Example. There is a metrizable non-archimedean space which has a non-σ-discrete antichain of open sets.

Proof. Consider the Baire space of weight κ, κ an uncountable cardinal, $X = B(\kappa) = \{x : x : \omega \to \kappa\}$. For $x = \langle x_n \rangle,\ y = \langle y_n \rangle \in B(\kappa)$, the distance

$d(x, y)$ is defined to be $\frac{1}{k}$, where $k \in \omega$ is least such that $x(k) \neq y(k)$. Let $x \in X$ such that $x(n) = 0$, $\forall n < \omega$. Let $V_{a,n} = \{y \in X : y|n = x|n, y(n) = \alpha\}$ for $\alpha < \kappa$ and $n < \omega$. Let $U_a = \bigcap\{V_{a,n} : n < \omega\}$, for $\alpha < \kappa$. For $\alpha, \beta < \kappa$, $U_\alpha \cap U_\beta = 0$. And for $\alpha < \kappa$, $x \in U_\alpha$. Therefore, the family $\{U_\alpha : \alpha < \kappa\}$ is a disjoint family, and any subfamily with more than one element is not discrete. ■

Now we answer Watson's question by proving

Theorem 11 *Any Souslin line (i.e., a ccc non-separable LOTS) is σ-dcc but not K_0'; any σ-dcc non-K_0' LOTS includes a Souslin line.*

Proof. A Souslin line is hereditarily Lindelöf, hence perfectly normal. It is not separable, but its discrete subspaces are countable, so it is not K_0'.

To prove the other half, it is convenient to note the following useful lemma:

Lemma 12 *If there is a generalized Souslin line, there is one with no K_0' intervals.*

Proof. The argument is from [T], Theorem 4.2. Let L be a generalized Souslin line and let $\mathcal{A}$ be a maximal antichain of open K_0 subspaces. Then $\overline{\bigcup \mathcal{A}}$ is K_0 so $L - \overline{\bigcup \mathcal{A}}$ is as desired. (Note it's a LOTS since it's perfectly normal non-archimedean [P].)

Now, given a non-K_0' σ-dcc LOTS L, we pass to the open subspace L' constructed in Lemma 12. Note L' is σ-dcc. Pick an $x \in L'$ and $\{x_n : n < \omega\}$ converging to x. Pick disjoint open intervals $I(n)$ containing x_n, $x \notin I_n$. Claim some I_n is ccc. Suppose not. Let $\{A_{n\alpha} : \alpha < \omega_1)$ be an antichain in I_n. Let $V_\alpha = \bigcup\{A_{n\alpha} : n < \omega\}$. $\{v_\alpha : \alpha < \omega_1\}$ is disjoint but is not σ-discrete since $x \in \overline{V_\alpha}$ for all α. Since I_n is ccc but not K_0', it is not separable and hence is a Souslin line. ■

3 Generalized Lusin lines

We next establish the remaining equivalence in Theorem 1 by proving

Lemma 13 *An archvillain includes a generalized Lusin line; a generalized Lusin line is an archvillain.*

Proof. An archvillain includes a generalized Souslin line, and by Lemma 12 includes an archvillain L without any K_0' intervals. Claim L is generalized Lusin. It suffices to prove nowhere dense subspaces of L are K_0'. but in [QT] we prove

Lemma 14 *Nowhere dense subspaces of archvillains are metrizable.*

But metrizable spaces are K_0'. ■

The other direction of Proposition 13 is almost immediate: by Lemma 4, it suffices to prove that if D is a relatively discrete subspace of a generalized Lusin line L, it is σ-discrete. Since L has no isolated points, $\overline{D}$ is nowhere dense, hence K_0'. But then so is D. ■

4 Archvillains with point-countable bases.

Since the property of having a point-countable base is hereditary, the reader will observe that our proofs of the various parts of Theorem 1 actually establish the equivalence of the first four clauses of Theorem 2 as well.

The remaining equivalence follows from Theorem 7. ■

To prove Theorem 3, we recall

Lemma 15 [*QT*]. *If it is consistent that there is a supercompact cardinal, then it is consistent that there is a model in which every archvillain has an archvillain subspace of size* $\aleph_1$.

The model is the Lévy or Mitchell collapse of the supercompact to $\aleph_2$. The subspace X is not K_0' so is not separable; archvillains are perfect LOTS, so first countable; taking a left-separated dense subspace of X of type ω_1, we obtain the required archvillain with a point-countable base. ■

On the other hand,

Theorem 16 *If it is consistent that there is an inaccessible cardinal, it is consistent there is an archvillain such that no archvillain subspace of it has a point-countable base.*

Proof. In [Q], from the assumption that there is an inaccessible cardinal, a model is constructed in which there is an $\aleph_2$-Souslin line which is an archvillain, but in which there are no archvillains of weight $\leq \aleph_1$. Since the line has hereditary Lindelöf number $\aleph_1$, a point-countable base of the subspace would have to be of cardinality $\leq \aleph_1$, so the subspace could not be an archvillain. ■

In conclusion, we thank the referee for greatly simplifying several of our proofs and finding errors in an earlier version.

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