REFLECTIONS ON DYADIC COMPACTA

Franklin D. Tall[1]

Abstract. Let M be an elementary submodel of the universe of sets, and $\langle X, \mathcal{T} \rangle$ a topological space in M. Let X_M be $X \cap M$ with topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. Let D be the two-point discrete space. Suppose the least cardinal κ of a basis for X_M is a member of M, and X_M is an uncountable continuous image of D^κ. Then $X = X_M$ if either $0^\#$ doesn't exist or κ is less than the first inaccessible cardinal. A corollary is that if G_M is a compact group and the least cardinal of a basis for G_M is in M, then $G = G_M$.

As usual, we formalize the notion of an elementary submodel of V, the universe of all sets, by considering elementary submodels of $H(\theta)$, the set of all sets of hereditary cardinality less than θ, for θ a "sufficiently large" regular cardinal. For a careful discussion of this point and for the basic facts about elementary submodels, we refer to [JW]. Now let $\langle X, \mathcal{T} \rangle$ be a topological space which is a member of M, an elementary submodel of some $H(\theta)$ as above. We define X_M to be $X \cap M$ with topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. The basic results about X_M are explored in [JT]. In [T] we initiated the study of when X_M characterizes X, and proved among other things that *if X_M is homeomorphic to the Cantor set, then* $X = X_M$. In [T_1] and [JT_1] this was extended to get

Theorem 1. *Let D be the two-point discrete space. Suppose X_M is homeomorphic to D^κ, where $\kappa \in M$ is less than the first inaccessible cardinal. Then $X = X_M$. If $0^\#$ does not exist, the inaccessibility restriction on κ can be removed.*

Here, the assertion that $0^\#$ does not exist is a consequence of Gödel's Axiom of Constructibility and is often stated equivalently as "The Covering Lemma for L". For more information on this, see [K], but in fact the only consequence of this assumption that we use is given in

Lemma 2 [KT]**.** *If $0^\#$ does not exist, then if M is an elementary submodel of some $H(\theta)$ and $|M| \geq \kappa$, then $\kappa \subseteq M$.*

We will implicitly assume all spaces are T_2. It is easy to see that if X_M is T_2, so is X [T]. In this note, we considerably improve Theorem 1 by weakening

[1]Research supported by NSERC Grant A-7354.

"homeomorphic" to "continuous". More precisely, a space is said to be a *dyadic compactum* if it is a continuous image of D^κ for some κ. We shall prove

Theorem 3. *Suppose X_M is an uncountable dyadic compactum and $w(X_M)$ (the least cardinal of a base for X_M) is less than the first inaccessible cardinal and is a member of M. Then $X_M = X$. If $0^\#$ does not exist, the inaccessibility restriction on w can be removed.*

An interesting consequence of this is

Corollary 4. *Suppose X_M is a compact topological group and $w(X_M) \in M$ is less than the first inaccessible cardinal. Then $X_M = X$. If $0^\#$ does not exist, the inaccessibility restriction on w can be removed.*

This is because compact groups are dyadic [Ku] and, by homogeneity and Baire category, uncountable.

Uncountability is needed, however, in Theorem 3, for consider $X = \omega_1 + 1$ and M a countable elementary submodel. Then X_M is homeomorphic to $\alpha + 1$ for some $\alpha < \omega_1$. $X_M \neq X$; claim X_M is dyadic. Let $f : \omega \to \alpha + 1$ be a bijection. Let $\{C_n\}_{n<\omega}$ be a sequence of disjoint copies of the Cantor set. The one-point compactification of $\bigcup\limits_{n<\omega} C_n$ is dyadic [E] and the map that sends C_n to $\{f(n)\}$ and the point at infinity to α is continuous, so $\alpha + 1$ is dyadic.

In [JT_1] we explored the general question of what additional properties on a compact X_M imply that $X = X_M$; the proof of Theorem 3 uses the methods of [T_1] and [JT_1] in conjunction with the technology for dyadic compacta developed by B. Efimov in a series of papers, in particular [E], [E_1], [E_2], and [E_3]. The following result is key.

Lemma 5 [E_3]**.** *If Y is dyadic and either Y is uncountable and $w(Y) = \aleph_0$, or else $cf(w(Y)) > \omega$, then Y maps onto $I^{w(Y)}$, where $I = [0, 1]$.*

Let $\kappa = w(X_M)$. Then, unless $\omega = cf(\kappa) < \kappa$, X_M maps onto I^κ. As in Efimov's work, the case when $cf(\kappa)$ is countable and less than κ requires considerably more care and effort, so we *first consider the case when $cf(\kappa)$ is uncountable.* (The case when $\kappa = \aleph_0$ is done in [T] and [JT_1].) As in [JT_1], under the hypotheses of Theorem 3, we can conclude that $2^\kappa \subseteq M$. Sketching the proof of that for the convenience of the reader, we first note that X is compact.

Lemma 6 [Ju]**.** *If Y_M is compact, so is Y, and Y_M is a continuous image of Y under the map π defined by $\pi^{-1}(y) = \bigcap\{U \in \mathcal{T} \cap M : y \in U\}$.*

This gives us a map from X onto I^κ, whence by elementarity we get a map from X_M onto $(I^\kappa)_M$ and indeed one onto I_M. Since I_M is dense in I, the compactness of I_M implies $I = I_M$. It follows, again by compactness, that $(I^\kappa)_M = I^{\kappa\cap M}$. We need only prove that $\kappa \subseteq M$ to conclude that $I^{\kappa\cap M} = I^\kappa$ and hence $2^\kappa \subseteq M$. This can be proved by induction for κ less than the first inaccessible [T_1], [JT_1]. It will also follow from Lemma 2 if we assume $0^\#$ doesn't exist, for if not, repeating the proof in [JT_1] of Theorem 1, there is a least $\alpha < \kappa$ such that $\alpha \subseteq M$ but $\notin M$. Since $|M| \geq |I^{\kappa \cap M}| \geq 2^{\kappa \cap M}$, $|M| \geq 2^{|\alpha|} \geq |\alpha|^+$. Then $|\alpha|^+ \subseteq M$, so $\alpha \in M$. Once we have $2^\kappa \subseteq M$, by Theorem 8 below we will get $X = X_M$. To establish Theorem 8 we need:

Lemma 7 [JT_1]**.** *Suppose Y_M is compact, $\chi(Y) \leq \lambda$, and $\lambda \subseteq M$. Then $Y_M = Y$.*

To prove this, one observes that the map π defined above in this case is one-to-one and hence is the identity homeomorphism.

We will use Lemma 7 in proving the following generalization of Theorem 5.1 of [JT_1], where the case of λ countable was proved.

Theorem 8. *Suppose Y_M is compact and $\chi(Y_M) \leq \lambda$. If $2^\lambda \subseteq M$, then $Y = Y_M$.*

Proof. By Lemma 7, it suffices to prove $\chi(Y) \leq \lambda$. Suppose not. Then by, for example [D], Y has a subspace Z of size λ^+ such that $\chi(Z) > \lambda$. By elementarity, since $2^\lambda \subseteq M$ implies $\lambda \in M$, we may take $Z \in M$. As in [JT_1], it suffices to show $\overline{Z} = \overline{Z}_M$ to obtain a contradiction, since $\chi(\overline{Z}_M) \leq \chi(Y_M) \leq \lambda$. Without loss of generality then, we may take $Y = \overline{Z}$.

Recall $t(Y)$ is the least cardinal τ such that whenever $y \in \overline{A} \subseteq Y$, there is a $B \subseteq A$, $|B| \leq \tau$, such that $y \in \overline{B}$. Again the case of λ countable in the following result was proved in [JT_1].

Lemma 9. *Suppose Y_M is compact and $t(Y_M) \leq \lambda$, where $\lambda^+ \subseteq M$. Then $t(Y) \leq \lambda$.*

The proof is as in [JT_1], using elementarity and the characterization of t in compact spaces in terms of free sequences. We can now prove Theorem 8 as in [JT_1]. For since $2^\lambda \subseteq M$, Lemma 9 applies, so $Y = \overline{Z} = \bigcup\{\overline{E} : E \in [Z]^\lambda\}$. By elementarity, since $Z \cap M = Z$, $Y_M = \bigcup\{\overline{E}_M : E \in [Z]^\lambda \cap M\}$. It suffices to show $\overline{E}_M = \overline{E}$, for then $Y_M = \bigcup\{\overline{E} : E \in [Z]^\lambda \cap M\}$. But $Z \in M$ and $\lambda \in M$, so $[Z]^\lambda \in M$. $|[Z]^\lambda| \leq (\lambda^+)^\lambda \leq 2^\lambda$, so $[Z]^\lambda \subseteq M$ so $[Z]^\lambda \cap M = [Z]^\lambda$ and then $Y_M = Y$.

To prove $\overline{E}_M = \overline{E}$, we need to generalize yet another result from [JT_1] by proving

Lemma 10. *Suppose Y_M is compact and $d(Y_M) \leq \lambda$ (where d of a space Z is the least cardinal of a dense subspace of Z). If $2^\lambda \subseteq M$, then $Y = Y_M$.*

Proof. As in [JT_1], the proof divides into two cases. First assume Y maps onto I^{2^λ}. Then, as usual, we conclude $2^{2^\lambda} \subseteq M$. Since $d(Y_M) \leq \lambda$, $w(Y_M) \leq 2^\lambda$. Since $2^{2^\lambda} \subseteq M$, as in [T] we conclude that Y has no left- or right-separated subspace of size $(2^\lambda)^+$ since Y_M doesn't, so $|Y| \leq 2^{2^\lambda}$. But then by compactness, $w(Y) \leq 2^{2^\lambda}$, whence by Lemma 7, $Y = Y_M$.

The other case is when Y does not map onto I^{2^λ}. We need a result of Šapirov'skiĭ, proved e.g. in [J]. Let $\rho(Y)$ be the number of regular open subsets of Y.

Lemma 11. *If Y is compact and does not map onto I^{κ^+}, then $\rho(Y) \le \kappa^{c(X)}$.*

Dyadic compacta satisfy the countable chain condition, i.e. $c(Y_M) = \aleph_0$, since powers of D do. Since $\omega_1 \subseteq M$, by elementarity Y also satisfies the countable chain condition. Since Y doesn't map onto I^{2^λ}, it certainly doesn't map onto $I^{(2^\lambda)^+}$, so $\rho(Y) \le (2^\lambda)^{\aleph_0} = 2^\lambda$. But then $w(Y) \le 2^\lambda$, so by Lemma 7, we are done with the proof of Lemma 10.

This completes the proof of Theorem 3 for the case when X_M has countable weight or weight of uncountable cofinality. In dealing with the case of $\kappa > cf(\kappa) = \omega$, where $\kappa = w(X)$, we are handicapped by the fact that X_M and hence X may not be assumed to map onto $D^{w(X_M)}$. However we will still be able to use the ideas of the uncountable cofinality proof, thanks to Efimov's analysis of dyadic spaces with countably cofinal uncountable weight. Efimov proved:

Lemma 12 [E$_2$]**.** *If $w(X) > cf(\omega(X)) = \omega$, where X is dyadic, then either X maps onto $I^{w(x)}$ or $X = \bigcup_{n<\omega} F_n$, where each F_n is closed, $w(F_n) < w(X)$, and the F_n's are increasing via inclusion.*

Note that we may without loss of generality assume each $w(F_n)$ is uncountable. To see this, recall

Lemma 13 [A]**.** *If Y is compact, $Y = \bigcup\{A_s : s \in S\}$, $w(A_s) \le \lambda \ge \aleph_0$ for all $s \in S$, and $|S| \le \lambda$, then $w(Y) \le \lambda$.*

Thus a compact space which is the union of countably many subspaces with a countable base itself has a countable base.

The first alternative given by Lemma 12 is dealt with exactly as for uncountable cofinality, so we consider the second alternative. Before dealing with dyadic compacta in general, it is interesting to note that we already have enough information to deduce Corollary 4, because we can show the second alternative cannot occur for compact groups. For suppose it did. By Lemma 12 and the Baire Category Theorem, there is a closed $F \subseteq G$ with $\operatorname{int} F \neq 0$ and $w(F) < w(G)$. Let $g \in \operatorname{int} F$. Then $\chi(g) \le w(F)$. Groups are homogeneous, so $\chi(G) = \chi(g)$. But in his survey [C], Comfort notes (p. 1158), that for a compact group G, $\pi\chi(G) = w(G)$. ($\pi\chi(x) = \min\{\lambda$: there is a collection $\mathcal{U}$ of open sets, $|\mathcal{U}| = \lambda$, such that every open set containing x includes a member of $\mathcal{U}\}$. $\pi\chi(X) = \sup\{\pi\chi(x) : x \in X\}$. $\mathcal{U}$ is called a *local π-base* for x. Since $\pi\chi(G) \le \chi(G) \le w(G)$, $\chi(G) = w(G)$, which yields a contradiction.

Returning to the countable cofinality case of Theorem 3, we claim that, without loss of generality, we may assume that

$$(*) \text{ for the } F_n\text{'s of Lemma 12, } \sum_{n<\omega} w(\overline{\operatorname{int} F_n}) = w(X) .$$

The advantage of working with the $\overline{\operatorname{int} F_n}$'s is that *closures of open subsets of dyadic compacta are themselves dyadic compacta* [E]. On the way to proving we can assume (∗), note that by the Baire Category Theorem, some F_n, hence all F_n from some point on, has non-empty interior. Thus without loss of generality, we may assume each of the F_n's has non-empty interior. Claim then $U = \bigcup_{n<\omega} \operatorname{int} F_n$ is dense in X. If not, there is a non-empty open $V \subseteq X - U$. Then $V = \bigcup_{n<\omega} F_n \cap V$. Again by Baire Category, for some n, $F_n \cap V$ has non-empty interior in V and hence in X. But $\operatorname{int}(F_n \cap V) \subseteq \operatorname{int} F_n \subseteq X - V$, contradiction. Now *$\pi$-weight = weight for dyadic compacta* [P], so it suffices to show $\sum_{n<\omega} \pi(\overline{\operatorname{int} F_n}) = \pi(X)$. ($\pi(X) = \min\{\lambda$: there is a collection $\mathcal{U}$ of non-empty open sets, $|\mathcal{U}| = \lambda$, such that each non-empty open set includes a member of $\mathcal{U}\}$.) But $\bigcup_{n<\omega} \overline{\operatorname{int} F_n}$ is dense in X, so $\pi\Big(\bigcup_{n<\omega} \overline{\operatorname{int} F_n}\Big) = \pi(X)$. Similarly $\pi\Big(\bigcup_{n<\omega} \operatorname{int} F_n\Big) = \pi\Big(\bigcup_{n<\omega} \overline{\operatorname{int} F_n}\Big)$. But $\pi\Big(\bigcup_{n<\omega} \operatorname{int} F_n\Big) \le \sum_{n<\omega} \pi(\operatorname{int} F_n)$, so

$$w(X) = \pi(X) \le \sum_{n<\omega} \pi(\operatorname{int} F_n) = \sum_{n<\omega} \pi(\overline{\operatorname{int} F_n}) = \sum_{n<\omega} w(\overline{\operatorname{int} F_n}) \le w(X)\ .$$

In order to make use of the uncountably cofinal case in finishing off the countably cofinal case, we need the following result:

Lemma 14. *Suppose X is a dyadic compactum and $\aleph_0 < w(X) = \sum_{n<\omega} w(X_n)$, where each X_n is a dyadic compactum included in X and $\aleph_0 < w(X_n) < w(X)$. Then there are dyadic compacta $X'_n \subseteq X$, $n < \omega$, such that $w(X) = \sum_{n<\omega} w(X'_n)$ and for each n, $cf(w(X'_n) > \omega$.*

Proof. By induction. This is clear for $w(X) = \aleph_\omega$. Assume true for each dyadic compactum with uncountable weight smaller than X. Then $w(X) = \sum_{n<\omega}\sum_{k<\omega} w(X_{nk})$, where each $cf(w(X_{nk})) > \omega$.

Assume then that there are dyadic compacta H_n, $n < \omega$, such that $H_n \subseteq X_M$, $cf(w(H_n)) > \omega$ and $w(X_M) = \sum_{n<\omega} w(H_n)$. Since X maps onto X_M, for each n, some closed subset of X maps onto H_n. But by Lemma 5, H_n maps onto $I^{w(H_n)}$. So for each n, some closed subset of X maps onto $I^{w(H_n)}$. By the usual argument, it follows that if either $0^\#$ does not exist or $w(X_M)$ is less than the first inaccessible, then $2^{w(H_n)} \subseteq M$. Thus $\sum_{n<\omega} 2^{w(H_n)} \subseteq M$, and in particular, $w(X_M) \subseteq M$.

With this preliminary work done, we now can start attacking the countable cofinality case of Theorem 3. There are two subcases to consider. First, suppose that $\kappa = w(X_M)$ is not a strong limit, i.e. for some $\lambda < \kappa$, $2^\lambda \ge \kappa$. Then $2^\kappa = \kappa^{cf(\kappa)} = \kappa^{\aleph_0} \le (2^\lambda)^{\aleph_0} = 2^\lambda$. By Theorem 8 or Lemma 10, we will be able to conclude that $X = X_M$ once we prove that $2^\lambda \subseteq M$. In order to obtain that, it will as usual suffice to prove that a closed subspace of X maps onto I^λ. That will follow by Lemma 6 from X_M having a closed subspace mapping onto I^λ. But we have that from Lemmas 12 and 14.

Now we will consider the case when $\kappa = w(X_M)$ is a strong limit cardinal of countable cofinality. We may assume X_M does not map onto I^κ, for if it did, our proof of the case when $\kappa = \aleph_0$ or is not countably cofinal would work. Let

$$S_\kappa = \{x \in X : \pi\chi(x) < \kappa\} .$$

Lemma 15. *If X is compact and does not map onto I^κ, then S_κ is G_δ-dense in X, i.e. S_κ meets every non-empty G_δ in X.*

This is proved in [J, 3.20] for the case of open sets rather than G_δ's but the only use of openness is that every non-empty open set includes a non-empty closed G_δ. But the same is true for non-empty G_δ's.

Since $\kappa \in M$ and $cf(\kappa) = \omega \subseteq M$, there is a sequence in M of smaller cardinals in M, $\{\kappa_n\}_{n<\omega}$, cofinal in κ. Let

$$K_n = \{x : \pi\chi(x) \leq \kappa_n\} .$$

Then $X = \bigcup_{n<\omega} \overline{K_n}$. For suppose not. Then $G = \bigcap_{n<\omega} (X - \overline{K_n}) \neq 0$, so $G \cap S_\kappa \neq 0$, contradiction.

Since, as noted after Lemma 13, $\kappa \subseteq M$, we can get $X \subseteq M$ if we can prove $|X| \leq \kappa$. That will follow if each $|\overline{K_n}| \leq \kappa$. Once we have $X \subseteq M$, we will have $X = X_M$, since X_M is a weaker T_2 topology on the compact space X. We will show that in fact each $|\overline{K_n}| < \kappa$ by showing that $\overline{K_n} = (\overline{K_n})_M \subseteq X_M$, and that $|(\overline{K_n})_M| < \kappa$. Note that $\overline{K_n} \in M$ since it is definable from X and κ_n. In order to prove that $\overline{K_n} = (\overline{K_n})_M$, we first establish two general results about the elementary submodel topology.

Lemma 16. *Suppose Y is a topological space in M. If $\pi\chi(y, Y) < \lambda$, for each $y \in Z \subseteq Y$, where Z and $\lambda \in M$, then also $\pi\chi(y, Y_M) < \lambda$, for all $y \in Z_M$.*

Proof. Note Z_M is a subspace of Y_M. Let $y \in Z_M$. There is a local π-base $\mathcal{P}$ for y in Y, with $|\mathcal{P}| < \lambda$. We may take $\mathcal{P} \in M$. Then claim $\{P \cap M : P \in \mathcal{P}\}$ is a local π-base for y in Y_M. For given a basic open $U \cap M$, $U \in M$, U open in Y with $y \in U \cap M$, there is a $P \in \mathcal{P}$ with $P \subseteq U$. By elementarity, we may take $P \in M$. Then $P \cap M \subseteq U \cap M$ as required.

Lemma 17. *If $K \in M$, K a subset of a space $Y \in M$, then $\overline{K}_M = \overline{K \cap M} = \overline{K_M}$ where the second and third closures are taken in Y_M.*

Proof. The last equality is clear. To prove the first, $y \in \overline{K}_M$ if and only if $y \in \overline{K} \cap M$ if and only if $y \in M$ and every open set about y in X meets K. But by elementarity, since y and $K \in M$, that's if and only if every open set containing y which lies in M meets K in M, which is if and only if $y \in \overline{K \cap M}$ in X_M.

Thus, getting back to the K_n's, we have $(\overline{K_n})_M = \overline{K_n \cap M}$. Now since X_M is compact, so is $(\overline{K_n})_M$. We will show that $|(\overline{K_n})_M| < \kappa$, for then, since κ is strong limit and is included in M, by Lemma 10 it will follow that $(\overline{K_n})_M = \overline{K_n}$, as required.

Again, we need a result of Efimov:

Lemma 18 [E_1]. *If Y is a dyadic bicompactum and $Z \subseteq Y$ and $\pi\chi(z, Y) \leq \lambda \geq \aleph_0$ for all $z \in Z$, then $w(\overline{Z}) \leq \lambda$.*

Let $I_n = \{x \in K_n \cap M : \pi\chi(x, X_M) < \aleph_0\}$. Then each point in I_n is isolated, so I_n is countable since, as mentioned before, X_M satisfies the countable chain condition. Taking closures in X_M, $\overline{K_n \cap M} = \overline{(K_n - I_n) \cap M} \cup \overline{I_n}$. By Lemma 18, the weight of the first term is $\leq \kappa_n$; on the other hand, the weight of the second is $\leq 2^{\aleph_0}$ since it is separable. By Lemma 13 then, $w(\overline{K_n \cap M}) \leq \kappa_n + 2^{\aleph_0} < \kappa$. Therefore $|\overline{K_n \cap M}| < \kappa$, since κ is strong limit.

By Lemma 17, this completes the proof of Theorem 3.

Remarks. It perhaps would be more natural to simply assume that some D^κ maps onto X_M, rather than $w(X_M) \in M$, but I do not see how to prove this. However, note that if $w(X_M)$ is definable, e.g. $\aleph_1$, $\aleph_{753}$, etc., then $w(X_M) \in M$. Some restriction on κ is required in Theorem 3, since in [T_1] I showed that if there is a *2-huge* cardinal, there are κ and M such that $(D^\kappa)_M$ is compact but not equal to D^κ.

After this paper was completed, K. Kunen improved Theorem 1, replacing "inaccessible" by "measurable". Since the non-existence of $0^\#$ implies there are no measurable cardinals, the final sentence of the theorem is rendered superfluous. Since the cardinality restrictions in Theorem 3 and Corollary 4 depend only on those of Theorem 1, they can be improved correspondingly.

Finally, I would like to thank Ofélia Alas and Lúcia Junqueira for catching errors in an earlier version of this paper.

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Franklin D. Tall, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3 CANADA
E-mail address: tall@math.utoronto.ca