A LINDELÖF SPACE WITH NO LINDELÖF SUBSPACE OF SIZE $\aleph_1$

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Abstract: A consistent example of an uncountable Lindelöf T_3 (and hence normal) space with no Lindelöf subspace of size $\aleph_1$ is constructed. It remains unsolved whether extra set-theoretic assumptions are necessary for the existence of such a space. However, our space has size $\aleph_2$ and is a P-space, i.e., G_δ's are open, and for such spaces extra set-theoretic assumptions turn out to be necessary.

1. Introduction.

The question of whether an uncountable T_2 space such as in the title exists is due to A. Hajnal and I. Juhász and has been open for almost thirty years. It first appeared in print in the list of problems in [R]. Hajnal and Juhász asked it after proving that under CH (the continuum hypothesis) there is no such compact T_2 example ([HJ]). The problem is interesting because all the other familiar properties equivalent to second countability in the class of metrizable spaces do "reflect" down to $\aleph_1$ (see [BT]). Until now, the only counterexample known for Lindelöfness failed even to be Hausdorff (see [BT]). Our example will be a subspace of the countable box topology on the product of $\aleph_2$ copies of the two-point discrete space, and so will be zero-dimensional. It is still open whether a T_2 example can be constructed in ZFC. See [BT] for an assortment of results asserting that there are no such examples if various topological and set-theoretic conditions are assumed.
Our example will be constructed by countably closed forcing. It in fact will be a *P-space*, i.e., G_δ's are open. This simplifies the task of avoiding Lindelöf subspaces of size $\aleph_1$ to that of just avoiding convergent ω_1-sequences. Notice that we speak of "avoiding" Lindelóf subspaces rather than "killing" them. That latter approach, while natural - do an iteration of length ω_2; each subset of size $\aleph_1$ appears at an initial stage; kill it by the next step of the forcing - will not work, since countably closed forcing preserves Lindelöfness for spaces of size $\aleph_1$ (see [BT]).
We also show that our example cannot be constructed in ZFC, since assuming for example the well-known combinatorial principle P_1 (see section 2 for the statement) which is a weak version of a generalized Martin's axiom (see [W]), we prove that every Lindelöf T_2 P-space of size $\aleph_2$ has a Lindelöf subspace of size $\aleph_1$.
Before moving on to the forcing construction of our space in section 3, we first do some topology in section 2, to verify our assertion that we need only avoid convergent ω_1-sequences, and to prove the result assuming P_1. In the last section 4, we include some remarks and questions related to our result. The set-theoretic terminology used in this paper is standard; for example if A, B are sets of ordinals we write $A < B$ to indicate that $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$. Other unexplained symbols or terms can be found in [K]. The terminology concerning general topology

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that is used in this paper follows [E]. We also use the powerful method of *elementary submodels* which is well-presented in [JuW] chapter 24.

2. Topological results.

We say that a transfinite sequence $\{x_\xi : \xi < \omega_1\}$ converges to a point x (and call it an ω_1-convergent sequence) whenever every open neighbourhood of x contains all but countably many points x_ξ. For example, if the character of a point x is exactly $\aleph_1$ and the point is not a G_δ-set, then by picking $x_\alpha \neq x$ from the intersection of the first α basic open sets, we can construct an ω_1-convergent sequence.
We will work with subspaces of 2^{ω_2} with the countable box topology which we denote by $\mathcal{T}$, and which is generated by a basis of sets of the form

$$[\sigma] = \{f \in 2^{\omega_2} : \sigma \subseteq f\},$$

for σ a function with a countable domain included in ω_2 and range 2. Note that the intersection of two sets of the form $[\sigma]$ is empty or again of the same form. Thus the family above is a basis for a topology. Let us note some simple facts.

FACT 1. $(2^{\omega_2}, \mathcal{T})$ *is a zero-dimensional, Hausdorff and hence regular space with a countably closed basis, i.e., the intersection of countably many basic open sets is open.*

PROOF: We leave it to the reader.

Note that an ω_1-convergent sequence together with its limit forms a Lindelöf subspace of size $\aleph_1$. Thus to obtain a space with no Lindelöf subspaces of size $\aleph_1$ we need to at least eliminate all ω_1-convergent sequences. As the following fact shows, sometimes that is all we need to do.

FACT 2. *Let X be a Lindelöf, Hausdorff space with a countably closed basis. Then Lindelöf subspaces of X are closed. Furthermore, if a subspace of size $\aleph_1$ is Lindelöf, then it includes a convergent ω_1-sequence.*

PROOF: Suppose $F \subseteq X$ is Lindelöf. Given $x \in X - F$, for each $y \in F$ let U_y, V_y be disjoint basic open sets containing x and y respectively. Then there exists $\{y_n : n \in \omega\}$ such that $\{V_{y_n} : n \in \omega\}$ covers F. Then $\bigcap\{U_{y_n} : n \in \omega\}$ is an open set containing x and disjoint from F. Thus $X - F$ is open.
Next we claim that if F is a Lindelöf subspace of X of size $\aleph_1$ and $p \in F$, then either p is isolated in F or $\chi(p, F) = \aleph_1$. As intersections of countably many basic open sets are open, there is no nonisolated $p \in F$ with $\chi(p, F) = \aleph_0$. So to see that either p is isolated in F or $\chi(p, F) = \aleph_1$, it is enough to prove the following

CLAIM: $\chi(p, F) \leq \aleph_1$.
PROOF: Let $\{V_\alpha : \alpha < \omega_1\}$ be a family of basic open sets containing p such that for each $q \in F - \{p\}$ there are $\alpha \in \omega_1$ and a basic $U(\alpha(q), q)$ such that

$$V_\alpha \cap U(\alpha(q), q) = \emptyset$$

and $q \in U(\alpha(q), q)$. We will show that $\{W_\beta : \beta < \omega_1\}$ forms a basis for p in F, where $W_\beta = \bigcap\{V_\alpha : \alpha < \beta\}$. First note that these sets are open as countable intersections of basic open sets. Then choose any open V containing p. $F - V$ is closed, hence Lindelöf and covered by $\{U(\alpha(q), q) : q \in F - V\}$, so choose a countable subcover $\{U(\alpha(q_n), q_n) : n \in \omega\}$. If $\beta = \sup\{\alpha(q_n) : n \in \omega\}$ then

$$W_\beta \cap \bigcup\{U(\alpha(q_n), q_n) : n \in \omega\} = \emptyset$$

which gives that $W_\beta \subseteq V$, completing the proof of the fact that $\{W_\beta : \beta < \omega_1\}$ forms a basis for p in F and completing the proof of the claim.

Now, since F is Lindelöf and uncountable, it cannot be discrete, so some point has character $\aleph_1$ (and that point cannot be G_δ, because it would be isolated) and hence is a limit of a convergent ω_1-sequence. Since an ω_1-sequence which converges in a subspace converges in the whole space, we have completed the proof of Fact 2, and have:

FACT 3. *If a subspace X of 2^{ω_2} with the topology $\mathcal{T}$ is Lindelöf and has no ω_1-convergent sequences then it has no Lindelöf subspace of size $\aleph_1$.*

Remark: Fact 2 and 3 are straightforward generalizations to Lindelöf P-spaces of standard facts about compact Hausdorff spaces; however our main result shows that another standard fact does *not* generalize, namely, that compact Hausdorff spaces of cardinality $< 2^{\aleph_1}$ are sequentially compact (see e.g. [vD]). Our Lindelöf P-space has cardinality $\aleph_2 < 2^{\aleph_2}$ and yet has no convergent ω_1-sequences. It also has no points of character less than $\aleph_2$, showing that the Čech-Pospíšil Theorem doesn't generalize.

Although we cannot prove the consistency of all Lindelöf (P-) spaces having a Lindelöf subspace of size $\aleph_1$, we can show this for Lindelöf P-spaces of size $\aleph_2$, at the cost however of assuming $2^{\aleph_1} > \aleph_2$. The following theorem can also be considered as a case when some generalization from compact to Lindelöf P-spaces does occur with respect to the existence of convergent sequences. For $i = 0, 1$, let P_i be the statement (see [W] 5.12.) :

Given a collection of fewer than $2^{\aleph_i}$ subsets of ω_i such that the intersection of any subcollection of cardinality less than $\aleph_i$ has cardinality $\aleph_i$, there is a subset $B \subseteq \omega_i$ of cardinality $\aleph_i$ such that for any member A of the collection we have $|B - A| < \aleph_i$.

By [vD], 6.2. P_0 is equivalent to the assertion that $2^{\aleph_0}$ is the minimal cardinal κ such that there is a compact space X with $\chi(X) = \kappa$ and a countable subset $A \subseteq X$ such thatthere is a cluster point x of A but no sequence of points of A which converges to x. Thus P_0 gives lots of convergent sequences in compact spaces of character less than 2^ω. The following is a generalization of this fact to Lindelöf P-spaces.

THEOREM 4. *It is consistent with CH plus $2^{\aleph_1} > \aleph_2$ that every T_2 Lindelöf P-space of cardinality $\aleph_2$ has an ω_1-convergent sequence and so has a Lindelöf subspace of size $\aleph_1$.*

PROOF: We use the fact that P_1 plus CH plus $2^{\aleph_1} > \aleph_2$ is consistent (see [K], p. 286). Thus it remains to prove that P_1 implies the statement of the theorem.

Let X be a T_2 Lindelöf P-space of cardinality $\aleph_2$. By a variation of the proof of Fact 2, it is routine to verify that $\chi(X) = \aleph_2^{\aleph_0} = \aleph_2 < 2^{\aleph_1}$ by CH.

Choose any $Y \subseteq X$ of cardinality $\aleph_1$. The Lindelöf property implies that Y has a complete accumulation point, call it x. Now the basic neighbourhoods of x obtained from the claim produce a countably closed uniform filter on Y of cardinality $< 2^{\aleph_1}$. Y can be identified with ω_1, thus P_1 can be applied. The B obtained from P_1 is an ω_1-convergent sequence; together with x it forms a Lindelöf subspace of X of cardinality $\aleph_1$.

3. The construction

We think of the collections of functions 2^A for $A \subseteq \omega_2$ as sets of branches through binary branching trees. We want to construct a set $X \subseteq 2^{\omega_2}$ and hence a subspace of $(2^{\omega_2}, \mathcal{T})$. We force with countably many countable pieces of functions so that X will consist of ω_2 of them. The requirements 1), 2), 5) on the forcing conditions and the forcing order are what one might expect. For the sake of the Lindelöf property of X, the projections $\pi_A(X)$ on the coordinates from a countable set $A \subseteq \omega_2$ need to be countable (thus the requirement 6) below). In fact, if $A \in [\omega_2]^{\omega_1} \cap V$, any ordering of A in the type of ω_1 will make $\pi_A(X)$ an ω_1-tree with no Aronszajn subtree (compare [JW]). Therefore we could call the combinatorial structure of X, *a forest with no Aronszajn tree.* Note that some kind of the requirement as in 4) below is necessary if the forcing is to satisfy the $\aleph_2$-c.c. The remaining technical requirements 3) and 7) are needed to avoid the ω_1-convergent sequences by "splitting" the points arbitrarily high. We force over a model of CH, which is needed in two of the lemmas.

DEFINITION 5. *The forcing P consists of conditions of the form $p = (a_p, b_p, F_p)$ which satisfy the following:*
1) $a_p \in [\omega_2]^\omega$, $b_p \in [\omega_2]^\omega$.
2) $F_p = \{f_p^\xi : \xi \in b_p\}$; $f_p^\xi : a_p \to 2$.
3) $\forall \xi \in b_p \forall \alpha \in a_p \exists \eta \in b_p - \alpha \quad f_p^\xi|(\alpha \cap a_p) = f_p^\eta|(\alpha \cap a_p)$.
4) $\forall \xi, \eta \in b_p,\ \xi \neq \eta\ \exists \alpha \in a_p \cap (\max(\xi, \eta) + 1) \quad f_p^\xi(\alpha) \neq f_p^\eta(\alpha)$.
The order is defined by $p \leq q$ if and only if the following are satisfied:
5) $a_p \supseteq a_q,\ b_p \supseteq b_q,\ \forall \xi \in b_q\ f_p^\xi|a_q = f_q^\xi$.
6)$\forall \xi \in b_p \exists \eta \in b_q \quad f_p^\xi|a_q = f_q^\eta$.
7) $\forall \xi, \eta \in b_q \forall \alpha \in a_q \quad f_q^\xi|(a_q \cap \alpha) = f_q^\eta|(a_q \cap \alpha) \ \Rightarrow f_p^\xi|(a_p \cap \alpha) = f_p^\eta|(a_p \cap \alpha)$.

LEMMA 6. *P is a σ-closed notion of forcing.*

PROOF: Suppose p_n is a decreasing sequence of conditions of P. Define $a_p = \bigcup\{a_{p_n} : n \in \omega\}$ and $b_p = \bigcup\{b_{p_n} : n \in \omega\}$. Finally for $\xi \in b_p$ put $f_p^\xi = \bigcup\{f_{p_n}^\xi : n \in \omega\}$. It is not difficult to check that p works as a lower bound of the sequence. However we may focus the attention of the reader on the fact that even though we can be building a complete binary tree while taking the sequence $(p_n : n \in \omega)$, at the limit we include (in the limiting condition) only countably many branches of this tree (to satisfy 1). This is similar to forcing a Kurepa tree with no Aronszajn subtree with countable conditions as in [T] and such a tree is related to the Lindelöf property as shown in [JW]. We will exploit this relationship below.

LEMMA 7. *For any $\alpha \in \omega_2$ the set $D_\alpha = \{p \in P : \ \alpha \in a_p\}$ is dense in P.*

PROOF: Take any $q \in P$. We show that there is an extension p of q such that $\alpha \in a_p$. We may without loss of generality assume that $\alpha \notin a_q$. If there is $\beta \in a_q$ such that $\beta > \alpha$, then we define $a_p = a_q \cup \{\alpha\}$, $b_p = b_q$ and $f_p^\xi = f_q^\xi \cup \{< \alpha, 0 >\}$ for all $\xi \in b_p$. It is easy to see that all the conditions 1) - 7) are satisfied. Now suppose that $a_p < \alpha$. In this case, fulfilling 3) will require more effort. Let $Q \subseteq 2^{[\alpha, \alpha+\omega)}$ denote the set of all functions with domain $[\alpha, \alpha+\omega)$, range 2 and such that the preimage of 1 is finite. Let 0^* denote the one of these functions which is constantly equal to 0. Let $\xi_0 > b_q, \alpha + \omega$. We define $a_p = a_q \cup [\alpha, \alpha + \omega)$, $b_p = b_q \cup [\xi_0, \xi_0 + \omega)$. To define F_p we need to define f_p^ξ for $\xi \in b_p$. If $\xi \in b_q$ then $f_p^\xi = f_q^\xi{}^\frown 0^*$. The f_p^ξ for $\xi \in b_p - b_q$ is just any enumeration of all functions of the form $f_q^\xi \frown g$ where $g \in Q - \{0^*\}$. It is easy to see that p as above works.

LEMMA 8. *For any $\xi \in \omega_2$, the set $E_\xi = \{p \in P : \ \xi < \sup(b_p)\}$ is dense in P.*

PROOF: Fix $q \in P$. We can add to b_q an interval $[\xi_0, \xi_0 + \omega)$ with $\xi_0 > \xi$ as in the second part of the proof of lemma 7, to obtain $p \in E_\xi$.

Given a generic set G for P, we define $B = \bigcup\{b_p : p \in G\}$ and $f^\xi = \bigcup\{f_p^\xi : p \in G\}$. Thus, by the previous lemmas, $X = \{f^\xi : \xi \in B\}$ is a family of $\aleph_2$ (in V, but below we will show that $\aleph_2$ is preserved) functions in 2^{ω_2}. Of course the f^ξ's depend on G; nevertheless we will skip the subscript G since we work with just one G. We will consider X as a subspace of $(2^{\omega_2}, \mathcal{T})$ where $\mathcal{T}$ is the countable box topology described in the previous section. As a subspace of a regular space, X is regular. We need to prove more topological properties of X in the generic extension.

LEMMA 9. *X is Lindelöf in V^P.*

PROOF: It will suffice to show that any cover of X by basic open sets $[\sigma]$ of $(2^{\omega_2}, \mathcal{T})$ includes a countable subcollection covering X. As the forcing is countably closed, these functions σ belong to the ground model; thus we can construct a decreasing sequence of conditions $(p_n : \ n \in \omega)$ such that for each $\xi \in b_{p_n}$ the condition p_{n+1} decides at least one basic open set $[\sigma_\xi]$ from the original cover which contains $\dot{f}^\xi$. We may without loss of generality assume by lemmas 6 and 7 that $dom(\sigma_\xi) \subseteq a_{p_{n+1}}$. Thus a $p \in P$ below each p_n as in the proof of the lemma 6, has the following properties:

a) $\forall \xi \in b_p \exists \sigma_\xi \ p \Vdash \dot{f}^\xi \in [\check{\sigma}_\xi] \in \dot{\mathcal{U}}$.
b) $dom(\sigma_\xi) \subseteq a_p$.

Now using 6) of the definition, of extension in P and the compatibility of the conditions of a generic set we conclude that p forces that $\{[\sigma_\xi] : \xi \in b_p\}$ is a cover of X. As this collection is countable, this establishes the Lindelöfness of X. Actually, this proof is similar to the proof that a Kurepa tree constructed in [T] has no Aronszajn subtrees.

According to fact 3 from the previous section we are left with showing that X has no ω_1-convergent sequence. For this, as well as for the $\aleph_2$-chain condition proof, the possibility of performing certain amalgamations is needed. The following technical lemma takes care of this. First we need the following:

DEFINITION 10. *We say that $p, q \in P$ are isomorphic if and only if*

$$a_p \cap a_q \cup \ b_p \cap b_q < a_p - a_q \cup \ b_p - b_q < a_q - a_p \cup \ b_q - b_p$$

and moreover there are order preserving bijections $i : a_p \to a_q$ and $j : b_p \to b_q$ such that i is the identity on $a_p \cap a_q$, j is the identity on $b_p \cap b_q$ and

$$f_q^{j(\xi)}(i(\alpha)) = f_p^\xi(\alpha)$$

for all $\xi \in b_p$ and $\alpha \in a_p$.

Although this use of "isomorphic" is reasonably standard, it should be noted that the relation is not transitive.

LEMMA 11.

a) Suppose $p, q \in P$ are isomorphic and there are $\alpha_0, \xi_0 \in \omega_2$ such that

i) $a_p \cup b_p < \alpha_0 < a_q - a_p \cup b_q - b_p < \xi_0$ and

ii) $a_q - a_p \neq \emptyset$.

Then there is an $s \leq p, q$.

b) Suppose in addition that

iii) $r \leq p$,

iv) $a_r, b_r \subseteq \alpha_0$,

v) $\delta \in b_p \cap b_q$, and that

vi) there are $\xi_1 \in b_r - b_p$ and $\eta_1 \in b_q - b_p$ such that

$$f_r^{\xi_1}|a_p = f_p^{\delta}|a_p \quad \textit{and} \quad f_q^{\eta_1}|(a_p \cap a_q) = f_p^{\delta}|(a_p \cap a_q).$$

Then $s \leq r, q$ may be chosen so that $f_s^{\xi_1}|a_q = f_q^{\eta_1}$.

PROOF: We prove part b) only, as the proof of part a) can be obtained by a simplification of the proof of part b).

Let $\Delta = a_p \cap a_q$ and $\Gamma = b_p \cap b_q$. We define the following amalgamation $s = (a_s, b_s, F_s)$:

$a_s = a_r \cup \{\alpha_0\} \cup a_q$.

$b_s = b_r \cup b_q \cup [\xi_0, \xi_0 + \omega)$.

$F_s = F_r^* \cup F_q^* \cup F \cup F_\Gamma$ where

$F_r^* = \{f_s^\xi : \xi \in b_r - b_q\}$,

$F_q^* = \{f_s^\xi : \xi \in b_q - b_r\}$,

$F_\Gamma = \{f_s^\xi : \xi \in \Gamma\}$,

$F = \{f_s^\xi : \xi \in [\xi_0, \xi_0 + \omega)\}$.

There is not much freedom in defining F_Γ; for $\xi \in \Gamma$ we choose

$f_s^\xi = f_r^\xi \cup \{< \alpha_0, 0 >\} \cup f_q^\xi$.

By the isomorphism of p and q, the defined sets are functions.

Now let us define F_r^*. Let $\sigma : F_r \to F_q$ be any function such that

a) $\sigma(f)|\Delta \subseteq f$,

b) if $f|\Delta = g|\Delta$, then $\sigma(f) = \sigma(g)$,

c) $\sigma(f_r^{\xi_1}) = f_q^{\eta_1}$.

σ can be defined because for any $\xi \in b_r$ there is $\xi' \in b_p$ such that $f_r^\xi|a_p = f_p^{\xi'}$ (by 6) for the extension $r \leq p$) and because for any $\xi' \in b_p$ there is $\eta \in b_r$ such that $f_q^\eta|\Delta = f_p^{\xi'}|\Delta$ (by the isomorphism of p and q). Also c) follows from the hypothesis about ξ_1 and η_1. Now, for $\xi \in b_r - b_q$ define

$f_s^\xi = f_r^\xi \cup \{< \alpha_0, 1 >\} \cup \sigma(f_r^\xi)$.

To define F_q^* we need the following

CLAIM: For every $\xi \in b_q - b_r$ there is at most one (or none) $\eta \in \Gamma$ such that $f_q^\eta|\Delta = f_q^\xi|\Delta$.

PROOF OF THE CLAIM: Suppose there are two such distinct η_1, η_2; Then by 4) of the definition of forcing (say, for q), there is an $\alpha \leq \max(\eta_1, \eta_2) + 1$ such that $f_q^{\eta_1}(\alpha) \neq f_q^{\eta_2}(\alpha)$. But, since $\Gamma < (a_p \cup a_q) - \Delta$ (see definition 10, of isomorphic conditions), we have that $\alpha \in \Delta$, a contradiction.

Now we define $\rho : F_q \to F_r$ such that

a) $\rho(f)|\Delta \subseteq f$,

b) if $f|\Delta = g|\Delta$, then $\rho(f) = \rho(g)$,

c) if there is $\eta \in \Gamma$ such that $f_q^\eta|\Delta = f|\Delta$, then $\rho(f) = f_r^\eta$,

d) there is $\xi \in b_p$ such that $\rho(f) = f_r^\xi$.

First, let us see that ρ as above can be defined. If the hypothesis of c) is satisfied, take $\rho(f)$ as in c) (by the claim it is uniquely defined), then d) is satisfied, a) is clear and b) follows from the claim. If the hypothesis of c) is not satisfied and $f = f_q^\xi$ we pick for example $f_p^{j(\xi)}$ where j is an isomorphism between b_q and b_p (by the assumption of the lemma) which does not affect the values on Δ. Then define $\rho(f) = f_r^{j(\xi)}$ and do it for other functions with the same restriction on Δ so that b) is satisfied.
Now for $\xi \in a_q - a_r$ define
$f_s^\xi = \rho(f_q^\xi) \cup \{<\alpha_0, 0>\} \cup f_q^\xi$.
Finally a definition of F is needed so that 3) of the definition of P is fulfilled: let
$F = \{f_r^\xi \cup \{<\alpha_0, 1>\} \cup f_q^\eta : f_q^\eta|\Delta = f_r^\xi|\Delta, \xi \in b_r - \Gamma, \eta \in b_q - \Gamma\} - F_r^*$.
and assume that it is somehow numbered by elements of $[\xi_0, \xi_0 + \omega)$.
This completes the definition of s. Let us now prove that $s \in P$ and that $s \leq r, q$. Clearly 1), 2), 5), 6) are satisfied by the construction. Now let us verify 7), for both of the extensions $s \leq r$ and $s \leq q$, as it will be used in the proofs of 3) and 4). Let ξ, η, α be as in 7).
Case 1: $\xi, \eta \in b_r, \alpha \in a_r$.
As $a_s - a_r > a_r$, there is nothing to prove.

Case 2: $\xi, \eta \in b_q - b_r, \alpha \in a_q - \Delta$.
As $f_q^\xi|\Delta = f_q^\eta|\Delta$, we have $\rho(f_q^\xi) = \rho(f_q^\eta)$, thus 7) is satisfied.

Case 3: $\xi, \eta \in b_q - b_r, \alpha \in \Delta$.
By the construction $\rho(f_q^\xi) = f_r^{\xi'}$, $\rho(f_q^\eta) = f_r^{\eta'}$, $f_s^\xi|\alpha = f_p^{\xi'}|\alpha$ and $f_s^\eta|\alpha = f_p^{\eta'}|\alpha$ for some $\xi', \eta' \in p$ (as defined in the definition of ρ, clause d)). Since $r \leq p$, by 7) for this extension we have that $f_r^{\xi'}$ and $f_r^{\eta'}$ agree on $a_r \cap \alpha$ and so f_s^ξ and f_s^η agree on $a_r \cap \alpha$ as well.

Case 4: $\eta \in \Gamma$, $\xi \in b_q - \Gamma$ and $\alpha \in \Delta$.
Similar to case 3).

Case 5: $\xi, \eta \in \Gamma$ and $\alpha \in a_q - \Delta$.
By 4) for q or p, this is impossible.

Case 6: $\eta \in \Gamma$, $\xi \in b_q - \Gamma$ and $\alpha \in a_q - \Delta$.
Since $f_p^\eta|\Delta = f_q^\xi|\Delta$ and $\eta \in \Gamma$, by clause c) of the definition of ρ we have that

$$f_s^\xi = f_r^\eta \cup \{<\alpha_0, 0>\} \cup f_q^\xi,$$

and by the definition of F_Γ we have that

$$f_s^\eta = f_r^\eta \cup \{<\alpha_0, 0>\} \cup f_q^\eta.$$

As $f_q^\xi|(a_q \cap \alpha) = f_q^\eta|(a_q \cap \alpha)$ we obtain that $f_s^\xi|(a_s \cap \alpha) = f_s^\eta|(a_q \cap \alpha)$ as required. This completes the proof of 7)

To prove 3) let us consider various cases. Note that we will frequently use 7) without mentioning it.

Case 1: $\xi \in \Gamma$, $\alpha \in a_r$ or $\alpha \in a_q$.
3) follows from 3) for r or q respectively (using 7)).

Case 2: $\xi \in \Gamma$, $\alpha = \alpha_0$.
Since $\alpha_0 < a_q - a_p \neq \emptyset$, pick any $\alpha \in a_q - a_r$ and apply 3) for a_q.

Case 3: $\xi \in b_r - b_q$.
If $\alpha \in a_r$, we are done by 3) for r. If $\alpha \in a_q - a_r$, consider $\sigma(f_r^\xi)$. By 3) for q there is $\eta \in b_q - \alpha$ such that $\sigma(f_r^\xi)|\alpha \cap a_q = f_q^\eta|\alpha \cap a_q$ and $\sigma(f_r^\xi) \neq f_q^\eta$. The first part of these equalities implies that in particular $f_q^\eta|\Delta = f_r^\xi|\Delta$. Now note that by the construction

$$f_s^\xi = f_r^\xi \cup \{<\alpha_0, 1>\} \cup \sigma(f_r^\xi).$$

Also the following holds:

$$f_r^\xi \cup \{<\alpha_0, 1>\} \cup f_q^\eta \in F \subseteq F_s,$$

as $f_r^\xi \cup \{<\alpha_0, 1>\} \cup f_q^\eta$ cannot be in F_r^* because only f_s^ξ is in F_r^* among the extensions of f_r^ξ. If $\alpha = \alpha_0$ we can use the above case for α's in $a_q - a_r$, which completes the proof of this case.

Case 4: $\xi \in b_q - b_p$.
We may without loss of generality assume that $\alpha \in a_q - a_p$. Then by 3) for q we are done.

Case 5: $\xi \in [\xi_0, \xi_0 + \omega)$ We have

$$f_s^\xi = f_r^{\xi'} \cup \{<\alpha_0, 1>\} \cup f_q^{\eta'},$$

where $f_q^{\eta'}|\Delta = f_r^{\xi'}|\Delta$ and $f_q^{\eta'} \neq \sigma(f_r^{\xi'})$. Again we may without loss of generality assume that $\alpha \in a_q - a_r$. By 3) and 4) for q we get the required $\eta \neq \eta'$, $\eta > \alpha$.

Thus, we have completed the proof of 3), so now let us proceed to the proof of 4). Again we consider several cases.

Case 1: $\{\xi, \eta\} \in [b_r]^2 \cup [b_q]^2$.
This follows from 4) for r or q respectively.

Case 2: $\xi \in (b_r - b_q) \cup [\xi_0, \xi_0 + \omega)$, $\eta \in b_q - b_r$.
$\alpha = \alpha_0$ works by the construction.

Case 3: $\xi \in b_r - b_q, \eta \in [\xi_0, \xi_0 + \omega)$.
As $f_s^\eta|a_r = f_r^{\eta'} \in F_r$, we can consider two subcases. First, if $\eta' \neq \eta$, then the required α is in a_r (by 4) for r) and since $a_r < [\xi_0, \xi_0 + \omega)$ we are done. Secondly if $\eta = \eta'$, then since $f_s^\eta \notin F_r^*$, that means that $f_s^\eta \neq f_s^{\eta'}$, which is enough as all elements of $[\xi_0, \xi_0 + \omega)$ are above a_s.

This completes the proof of 4) and the fact that $s \leq r, q$.

LEMMA 12. (CH) *P satisfies the $\aleph_2$-chain condition.*

Let $\mathcal{A}$ be a family of distinct conditions of P of cardinality $\aleph_2$. We need to find two compatible conditions. Using CH we can choose two isomorphic conditions of $\mathcal{A}$, say p, q such that there are $\alpha_0, \xi_0 \in \omega_2$ such that $a_p, b_p < \alpha_0 < \alpha_0 + \omega < a_q - a_p, b_q - b_p < \xi_0$. We have two cases. If $a_p - a_q \neq \emptyset$, then we can apply lemma 11 for $r = p$ to obtain the required $s \leq p, q$. Otherwise $a_p = a_q$. The amalgamation s is obtained by putting $a_s = a_p \cup [\alpha_0, \alpha_0 + \omega)$, $b_s = b_p \cup\ b_q \cup [\xi_0, \xi_0 + \omega)$ and by defining $f_s^\xi = f_p^\xi{}^\frown 0^*$ for $\xi \in b_p$, $f_s^\xi = f_q^\xi{}^\frown 1^*$ for $\xi \in b_q$, and choosing f^ξ's for $\xi \in [\xi_0, \xi_0 + \omega)$ so that $\{f^\xi : \ \xi \in [\xi_0, \xi_0 + \omega)\} = \{f^\xi{}^\frown f : \xi \in b_p,\ f \in Q - \{0^*\}\}$ where i^* is the function constantly equal to i and defined on $[\alpha_0, \alpha_0 + \omega)$ for $i = 0, 1$ and Q is the set of all functions defined on $[\alpha_0, \alpha_0 + \omega)$ with at most finitely many values different than 0. A simple argument shows that s works.

LEMMA 13. (CH) *P forces that X has no convergent ω_1-sequences in V^P.*

PROOF: Suppose there is an ω_1-sequence converging to a point $f^\delta \in X$ for some $\delta \in B$. Let p_0 be a condition of P which forces it. Let $\{\dot{\xi}_\theta : \theta < \omega_1\}$ be P-names for distinct indexes of points of the sequence. Let M be an elementary submodel of $H(\omega_3)$ of cardinality $\aleph_1$ which is closed under countable subsets (i.e., $[M]^\omega \subseteq M$; here we are using CH) and contains all relevant objects, in particular p_0, δ, $\{\dot{\xi}_\theta : \theta < \omega_1\}$. Note that $M \cap \omega_2$ is an ordinal in ω_2 of uncountable cofinality.
Using the density lemmas 7, 8 and 3) and 4) of the definition 5 of the forcing P we can find a condition $p_1 \leq p_0$, and an ordinal $\alpha_1 \in a_{p_1}$ such that $\alpha_1 > M \cap \omega_2$ and an ordinal $\eta_1 \in b_{p_1} - M$ such that

$$f^{\eta_1}_{p_1}|(a_p \cap \alpha_1) = f^{\delta}_{p_1}|(a_p \cap \alpha_1), \quad f^{\eta_1}_{p_1}(\alpha_1) \neq f^{\delta}_{p_1}(\alpha_1).$$

Since $p_1 \leq p_0$ forces that $\{f^{\dot{\xi}_\theta} : \theta < \omega_1\}$ is a sequence converging to f^δ, there is a $q \leq p_1$ and a $\theta_0 < \omega_1$ such that q forces that for $\theta > \theta_0$ we have

*)
$$f^{\dot{\xi}_\theta}(\alpha_1) = f^\delta(\alpha_1).$$

Let us see that the amalgamation lemma 11 implies that this is impossible.

CLAIM: There is a condition p in $M \cap P$ which is isomorphic to q and such that $a_q \cap (M \cap \omega_2) = a_q \cap a_p$ and $b_q \cap (M \cap \omega_2) = b_q \cap b_p$. Since $\delta \in a_q \cap (M \cap \omega_2)$, it will follow that $\delta \in a_p$.

Proof of the claim: This can be arranged using the elementarity of M and the closure of M under countable subsets.

Now, work in M. As p forces that $[f^\delta_p|a_p]$ is a basic neighbourhood of f^δ and $\{f^{\dot{\xi}_\theta} : \theta < \omega_1\}$ is an ω_1-sequence converging to f^δ, p forces that there is a $\xi_\theta \in \omega_2 - b_p$ and $\theta > \theta_0$ such that $f^{\xi_\theta}|a_p = f^\delta_p|a_p$. So let $r \leq p$ be a condition which decides this ξ_θ and such that $r \in M$ and $\xi_\theta \in b_r$. Since the cofinality of $M \cap \omega_2$ is ω_1 we can choose $\alpha_0 \in M$ such that $a_r, b_r \subseteq \alpha_0$.
Now we check that we can apply lemma 11 for $\xi_1 = \xi_\theta$. Since $f^{\eta_1}_{p_1}(\alpha_1 \cap a_{p_1}) = f^\delta_{p_1}(\alpha_1 \cap a_{p_1})$, by 7) of the definition of P, $f^{\eta_1}_q(\alpha_1 \cap a_{p_1}) = f^\delta_q(\alpha_1 \cap a_{p_1})$, but $a_p \cap a_q = a_q \cap (M \cap \omega_2) \subseteq \alpha_1$.
¿From lemma 11 then, we obtain $s \leq r, q$ such that $f^{\xi_\theta}_s|a_q = f^{\eta_1}_q$. Then s forces that

$$f^{\dot{\xi}_\theta}(\alpha_1) = f^{\eta_1}_q(\alpha_1) \neq f^\delta_q(\alpha_1) = f^\delta(\alpha_1),$$

which contradicts *).

Combining lemmas 6, 9, 12, 13 and fact 3 we obtain the following.

THEOREM 14. *It is consistent that there is an uncountable Lindelöf regular space with no Lindelöf subspace of size $\aleph_1$.*

4. Remarks

There are a number of difficult long-outstanding open problems concerning Lindelöfness, for example Arhangel'skiĭ's problem concerning the existence of a Lindelöf space with points G_δ of size bigger than $2^{\aleph_0}$ (see [A]). A breakthrough there was achieved by Shelah [S], who constructed a consistent example of size $\aleph_2 = (2^{\aleph_0})^+$ by countably closed forcing. Later I. Gorelic (in [G]) found an essential simplification of this construction which generalizes to any cardinal $\kappa = 2^{\aleph_1} \geq \aleph_2 = (2^{\aleph_0})^+$. However no better ZFC bound for cardinalities of Lindelöf spaces with points G_δ than a measurable cardinal is known.

Previous unsuccessful attempts to produce a Lindelöf space with no Lindelöf subspace of size $\aleph_1$ may perhaps have been too influenced by Shelah's space, in that they carried the excess baggage of "points G_δ". We go to the opposite extreme here by having G_δ's open, which simplifies the construction. Nonetheless, our method of obtaining the Lindelöfness of the whole space is similar to that of [S], although we originally extracted it while looking at an independent argument from [JW].

We do not know how to construct a consistent example of a Lindelöf space with points G_δ which does not include a Lindelöf subspace of size $\aleph_1$. In [BT], it is shown consistent with ZFC - modulo a huge cardinal - that no such first countable space of size $\aleph_2 = (2^{\aleph_0})^+$ exists. This is non-trivial in the T_1 case - it is not known whether the cardinality of Lindelöf, first countable T_1 spaces is bounded by $2^{\aleph_0}$. Allowing $\neg$CH, it is consistent from a weakly compact cardinal that no T_2 counterexample of character less than $2^{\aleph_0}$ exists ([BT]).

The question remains of whether an uncountable Lindelöf space with no Lindelöf subspace of size $\aleph_1$ exists in ZFC, and in particular whether there is such a Lindelöf P-space.

References:

[A] A. Arhangel'skiĭ; *On the cardinality of bicompacta satisfying the first axiom of countability*; Soviet Math. Dokl. 10, 1969, pp. 951 - 955.

[BT] J. E. Baumgartner, F. D. Tall; *Reflecting Lindelöfness*; Top. Appl. To appear.

[vD] E. Van Douwen; *The Integers and Topology*; in [KV] pp. 111 - 167.

[E] R. Engelking; *General Topology*; Helderman Verlag, Berlin 1989.

[G] I. Gorelic; *The Baire category and forcing large Lindelöf spaces with points G_δ*; Proc. Amer. Math. Soc. 118, 1993, No 2. pp. 603 -607.

[HJ] A. Hajnal, I. Juhász; *Remarks on the cardinality of compact spaces*; Proc. Amer. Math. Soc. 59, 1976, pp. 146 - 148.

[JW] I. Juhász, W. Weiss; *On a problem of Sikorski*; Fund. Math. 100, 1978, pp. 223 - 227.

[JuW] W. Just, M. Weese; *Discovering Modern Set-Theory. Set-theoretic tools for every mathematician.* Vol II. Graduate Studies in Mathematics. Vol 18. American Mathematical Society, providence, 1997.

[K] K.Kunen; *Set Theory; An Introduction to Independence Proofs*; North Holland, 1986.

[KV] K. Kunen, J. Vaughan eds. *Handbook of set-theoretic topology*; North Holland, 1980.

[R] M. E. Rudin; *Lectures on Set-Theoretic Topology*; American Mathematical Society, Providence, 1975.

[S] S. Shelah; *On some problems in general topology* pp. 91 - 101, in *Set Theory*; Boise, Idaho; 1992 - 1994; American Mathematical Society; Providence 1996.

[T] S.Todorcevic; *Trees, subtrees and order types*; Ann. Math. Logic 20, 1981, pp. 233-268.

[W] W. Weiss; *Versions of Martin's Axiom*; in [KV] pp. 827 - 886.

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