REFLECTING LINDELÖFNESS

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Abstract. We consider the question of whether uncountable Lindelöf spaces have Lindelöf subspaces of size $\aleph_1$. T_3 plus countability conditions yield affirmative results under CH, but T_1 need not in ZFC. We also obtain some positive results by collapsing large cardinals.

Among the classic global countability properties of topology, some – such as *countable weight* (having a countable base) – are inherited by arbitrary subspaces, while others – such as *separability* (having a countable dense subspace) – are not. Among the latter, however, one notes that e.g. open subspaces inherit separability and the *countable chain condition* (disjoint open collections are countable), while closed subspaces inherit *Lindelöfness* (every open cover has a countable subcover). A natural question, especially in light of recent interest by set-theoretic topologists in the general topic of reflection (see e.g. [D_3], [D_4], [DTW_1]) is whether these countability properties are necessarily inherited by some small but uncountable – say of size $\aleph_1$ – subspace. For separability, this is trivially true; for the countable chain condition it is true by a standard easy Löwenheim-Skolem argument, probably first used in the consistency proof for Martin's Axiom [ST]. The question for Lindelöfness, however, raised in 1975 and 1976 by Hajnal and Juhász [Ru], [HJ], remains unsolved. Here we contribute some partial results. We are grateful to the referee for many suggestions and some specific results credited below.

Hajnal and Juhász gave an affirmative answer under CH for compact T_2 spaces:

Theorem 1. *CH implies every uncountable compact T_2 space has a Lindelöf subspace of size $\aleph_1$.*

We extend this to k-spaces and prove similar results when various countability properties are imposed:

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Theorem 2. *CH implies an uncountable Lindelöf T_2 space has a Lindelöf subspace of size $\aleph_1$ provided any of the following conditions hold:*

a) *X is sequential,*
b) *X is k,*
c) *X is locally separable,*
d) *X has countable spread,*
e) *X has countable tightness.*

On the other hand, we have

Theorem 3. *There is an uncountable Lindelöf T_1 space with no Lindelöf subspace of size $\aleph_1$.*

We don't know if T_1 can be improved, but we will show

Theorem 4. *If it is consistent there is an uncountable Lindelöf T_i space, $i \leq 3$, with no Lindelöf subspace of size $\aleph_1$, there is one of size $\aleph_2$.*

If it is consistent that there are no T_i examples, $i \geq 2$, one would expect to prove this by collapsing a large cardinal and proving some preservation result. We in fact show that the collapse considered in [K] does preserve Lindelöfness, but the pickings are slim, since all we have been able to prove from this is

Theorem 5. *In Kunen's model [K] for (CH and) a saturated ideal from a huge cardinal, every first countable Lindelöf space of size $\aleph_2$ has a Lindelöf subspace of size $\aleph_1$.*

No separation axioms are assumed here, but the result is rather unsatisfactory since not even a consistent example of a first countable T_1 Lindelöf space of size $> 2^{\aleph_0}$ is known, while Arhangel'skiĭ's theorem [A_2] bounds the cardinality of first countable T_2 Lindelöf spaces by $2^{\aleph_0}$.

The first third of the paper is topological, using elementary cardinal function theory as in e.g. [H], but also uses some easy elementary submodel arguments. The second third contains our only example, as well as some simple forcing arguments. The final third is almost entirely set-theoretic; a large number of mainly folklore results relating elementary submodels and forcing are established and then they are applied to (a simplification of) Kunen's construction. Our notation is fairly standard; undefined topological terms can be found in [E].

Now for the proof of Theorem 2.

Proof.

a) It suffices to establish that every uncountable T_2 sequential space has an uncountable Lindelöf subspace of size $\leq 2^{\aleph_0}$. We can do this by taking a countably closed elementary submodel of size $\leq 2^{\aleph_0}$ of a sufficiently large $H(\theta)$, with M containing X and including an uncountable piece of it. Then $X \cap M$ is closed in X [JT] so is Lindelöf.

b) By Theorem 1, without loss of generality, compact subsets of X are countable. But then X k implies X is Fréchet, a fortiori sequential.

To prove the last three, it is convenient to establish the following lemma, which is due to the referee.

Lemma 6. *CH implies every uncountable separable Lindelöf T_2 space includes a Lindelöf subspace of size $\aleph_1$.*

From this, the remaining clauses follow swiftly. For **c)**, it suffices to note that if all countable sets had countable closure, X would be countable. For **d)**, if countable subsets have countable closure, then take a strictly increasing chain $\{S_\alpha\}_{\alpha<\omega_1}$ of countable closed sets. Then, by countable tightness, $S = \bigcup_{\alpha<\omega_1} S_\alpha$ is a closed and hence Lindelöf subspace of size $\aleph_1$. For **e)**, again if countable sets have countable closure, we can take a strictly increasing ω_1-sequence of countable closed sets. This yields a left-separated subspace of size $\aleph_1$, which by countable spread is (hereditarily) Lindelöf.

Proof of Lemma 6. If every point is a G_δ, we in fact can prove that $|X| \leq 2^{\aleph_0}$. Let D be countable dense in X. We will construct an injection from X into $\big(\mathcal{P}(D)\big)^{\aleph_0}$. For each $x \in X$, fix open $\{U(x,n)\}_{n<\omega}$ witnessing that x is a G_δ. Since for each x, n, $X - U(x,n)$ is Lindelöf, $\{U(y,m) : y \in X - U(x,n) \text{ and } x \notin U(y,m)\}$ has a countable subcollection covering $X - U(x,n)$. Pick some such subcollection and let $\{S_x(n,k) : k < \omega\}$ list the intersections with D of each member of it. Since x is not a member of $\overline{S_x(n,k) \cap (X - U(x,n))}$ for any n, but any y different from x is in some $\overline{S_x(n,k) \cap (X - U(x,n))}$ for some n, we see that the assignment of $\{S_x(n,k) : n,k < \omega\}$ to x is one-one.

Now we consider the case when some $x \in X$ is not a G_δ. By CH, there are only $\aleph_1$ possible intersections of open neighbourhoods of x with a countable dense set D. Let $\{U_\alpha : \alpha \in \omega_1\}$ enumerate enough such neighbourhoods to yield all such intersections cofinally often. Recursively pick $x_\alpha \in X - \{x\}$ and ordinals $\gamma_\alpha \geq \alpha$ so that $x_\alpha \in \operatorname{int} \overline{U}_\beta$ for all $\beta < \alpha_\alpha$. At limit γ, simply pick $\gamma = \sup_{\alpha<\gamma} \gamma_\alpha$ and take $x_\gamma \in \bigcap_{\beta<\gamma} \operatorname{int} \overline{U}_\beta$. For $\alpha = \delta + 1$, by Hausdorff pick γ_α sufficiently large such that $x_\delta \notin \overline{U_{\gamma_\alpha} \cap D}$, and again pick $x_\alpha \in \bigcap_{\beta\leq\alpha} \operatorname{int} \overline{U}_\beta$. We shall show that $\{x\} \cup \{x_\alpha : \alpha < \omega_1\}$ is Lindelöf by proving that any open set about x contains all but countably many x_α's. Let W be any open neighbourhood of X and suppose on the contrary that $\{x_\alpha : \alpha \in \omega_1\} - W$ is uncountable. Since X is Lindelöf, there is a complete accumulation point z of this set. Let U and V be disjoint open sets about x and z respectively. Let β be such that $U \cap D = U_\beta \cap D$. Take $\alpha > \beta$ sufficiently large such that $x_\alpha \in V$. But $x_\alpha \in \overline{U_\beta \cap D}$, contradiction.

Hajnal and Juhász used a similar idea in order to prove Theorem 1, showing

Theorem 7. *If X has a point of character $\aleph_1$ and pseudocharacter $\aleph_1$, then X has a convergent ω_1-sequence, and hence a Lindelöf subspace of size $\aleph_1$.*

This result yields an interesting connection between their problem and Arhangel'skiĭ's famous problem of whether there exist Lindelöf T_2 spaces of size $> 2^{\aleph_0}$ with all points G_δ:

Theorem 8. *If there is an uncountable Lindelöf space of character $\leq \aleph_1$ which does not have a Lindelöf subspace of size $\aleph_1$, then it is a Lindelöf space of size $\geq \aleph_2$ with all points G_δ.*

The proof is immediate. This yields the following corollary:

Corollary 9. *If it is consistent that there is a supercompact cardinal, it is consistent that CH holds, that $2^{\aleph_1}$ is arbitrarily large, and that every uncountable Lindelöf T_2 space of size $< 2^{\aleph_1}$ and character $\leq \aleph_1$ has a Lindelöf subspace of size $\aleph_1$.*

Proof. By [T3], the supercompact yields a model in which CH holds and Lindelöf spaces of size $< 2^{\aleph_1}$ but $\geq \aleph_2$ have non-G_δ points.

Recall a space is *scattered* if every subspace contains an isolated point. O.T. Alas (private communication) recently answered a question of ours by proving

Theorem 10. *Every scattered Lindelöf T_3 space includes a Lindelöf subspace of size $\aleph_1$.*

We do have a couple of results in which we do not assume such a strong condition as "scattered", or countability conditions.

Theorem 11. *Suppose $2^{2^{\aleph_1}} = \aleph_3$, X is Lindelöf T_3, and $|X| \geq \aleph_3$. Then X has an uncountable Lindelöf subspace of cardinality and weight $\leq \aleph_3$.*

Proof. Take $Y \subseteq X$, $|Y| = \aleph_1$. Then $\left|\overline{Y}\right| \leq 2^{2^{\aleph_1}}$ and the weight of $\overline{Y}$ is $\leq 2^{\aleph_1}$. As a closed subspace of a Lindelöf space, $\overline{Y}$ is Lindelöf.

Thus, if we could get Lindelöf subspaces of size $\aleph_1$ inside Lindelöf T_3 spaces of size and weight $\leq \aleph_3$, we could get them inside any Lindelöf T_3 space.

Although $\aleph_3$ is "far enough" for Lindelöf T_3 spaces, there is no reason to believe this is so for e.g. T_1 spaces, even the first countable ones. However we do have one result which gives some information without separation axioms, provided the space hereditarily satisfies a weak covering condition.

Definition. A space is *countably metacompact* if each countable open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ such that each point is in at most finitely members of $\mathcal{V}$.

Theorem 12. *Suppose X is Lindelöf, uncountable, and hereditarily countably metacompact. Then X has a Lindelöf subspace of cardinality $\leq \beth_\omega$ (= the limit of the cardinals $\beth_n$ where $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$).*

Definition. Let Y be a subset of a space X. $x \in X$ is a *complete accumulation point* of Y if every neighbourhood of x has $|Y|$ many points of Y.

Proof. Recursively construct $Y_n, n < \omega$ as follows. Let Y_0 be any subspace of X of cardinality $\aleph_1$. Let Y_{n+1} be formed by taking Y_n together with one complete accumulation point for each subset of Y_n of uncountably cofinality. Such a point exists by Lindelöfness. Let $Y = \bigcup_{n<\omega} Y_n$. Then $|Y| \leq \beth_\omega$ and each subset of Y of uncountable cofinality has a complete accumulation point. But Y is countably metacompact, and it is not difficult to prove that that plus subsets of uncountable cofinality having complete accumulation points implies Lindelöfness [M].

With the aid of large cardinals, we can sometimes get Lindelöf subspaces of size $\aleph_1$ even without CH.

Theorem 13. *Con (there is a supercompact cardinal) implies Con ($2^{\aleph_0} = \aleph_2$ and every uncountable Lindelöf space of character $\leq \aleph_1$ has a Lindelöf subspace of size $\aleph_1$.)*

Theorem 14. *Con (there is a weakly compact cardinal) implies Con ($2^{\aleph_0} = \aleph_2$ and every uncountable Lindelöf T_2 space of character $\leq \aleph_1$ has a Lindelöf subspace of size $\aleph_1$).*

Proofs. Dow [D_1, 3.3 and the proof of 5.2] has shown that the Mitchell collapse [Mi] preserves Lindelöfness, so Theorem 13 follows. Theorem 14 is proved in the same way (see [DTW_2]); one needs to observe that by Arhangel'skiĭ's theorem [A], the cardinality of the space is $\leq 2^{\aleph_1}$ which equals $2^{\aleph_0}$ in the Mitchell model, and that the Lindelöf property is Π^1_1. To see the latter, note that since $|X| \leq \aleph_2 < \aleph_\omega$, X being Lindelöf is equivalent to every subset of X having a complete accumulation point.

In the absence of CH it is not even known whether all compact T_2 spaces have Lindelöf subspaces of size $\leq 2^{\aleph_0}$. Juhász and Szentmiklóssy [JS] establish

Theorem 15. *Every compact T_2 space of uncountable tightness contains a convergent ω_1-sequence.*

It follows that

Theorem 16. *Every T_2 k-space of uncountable tightness has a Lindelöf subspace of size $\aleph_1$.*

Proof. We need to show some compact subspace K has uncountable tightness. Suppose not. Claim X has countable tightness. If not, then for some $Y \subseteq X$, $\bigcup\{\overline{Z} : Z \in [Y]^{\aleph_0}\}$ is not closed. Therefore for some compact K, $L = K \cap \bigcup\{\overline{Z} : Z \in [Y]^{\aleph_0}\} = \bigcup\{\overline{Z} \cap K : Z \in [Y]^{\aleph_0}\}$ is not closed, hence not closed in K. But K has countable tightness, so there is a countable $S \subseteq L$ such that $\overline{S} \cap (K - L) \neq 0$. But S is included in $\bigcup\limits_{n<\omega} \overline{Z}_n \cap K$ for some $\{Z_n\}_{n<\omega}$, $Z_n \in [Y]^{\aleph_0}$, so S is included in a single $\overline{Z} \cap K$, for some $Z \in [Y]^{\aleph_0}$, so $\overline{S} \subseteq L$, contradiction.

We also have

Theorem 17. *Adjoin any number of Cohen reals. Then uncountable Lindelöf T_2 k-spaces have uncountable Lindelöf subspaces of size $\leq 2^{\aleph_0}$.*

Proof. If X has uncountable tightness, we are done. Otherwise, every compact subspace has countable tightness. Juhász and Szentmiklóssy [JS] show that uncountable non-first countable compact T_2 spaces with countable tightness have convergent ω_1-sequences in the Cohen model, so the remaining two cases are when the space is first countable or all compact subspaces are countable.

The first case is immediate from Arhangel'skiĭ's theorem; the second follows from the proof of Theorem 2.

Theorem 18. *Adjoin any number of Cohen reals. Then uncountable compact T_2 spaces of countable tightness have uncountable compact subspaces of size $\leq 2^{\aleph_1}$.*

This is of course of most interest when say we adjoin $\aleph_2$ Cohen reals to a model in which $2^{\aleph_1} \leq \aleph_2$, so that $2^{\aleph_1} = 2^{\aleph_0} = \aleph_2$.

Proof. If countable subsets have countable closures, apply the proof of Theorem 2. Otherwise, apply the result of [D_1] that in the Cohen model, compact T_2 separable spaces have cardinality $\leq 2^{\aleph_1}$.

Assuming $\diamondsuit$, there is a compact T_2 space of countable tightness which has no compact subspace of size $2^{\aleph_0}$. It is Fedorčuk's compact S-space [F]. Every infinite closed subset of it has cardinality $> 2^{\aleph_0}$.

Similar results follow from PFA:

Theorem 19. *PFA implies every uncountable Lindelöf T_2 k-space has an uncountable Lindelöf subspace of size $\leq 2^{\aleph_0}$.*

Proof. By Juhász and Szentmiklóssy, we may assume X has countable tightness. By [B], X is sequential. By Theorem 2, we are done.

Theorem 20. *PFA implies every uncountable compact T_2 space of countable tightness has an uncountable compact subspace of size $\leq 2^{\aleph_0}$.*

Again, X is sequential, so as in the proof of Theorem 2a), the elementary submodel does the job.

Next we come to our only example – only T_1 – of a Lindelöf space with no Lindelöf subspace of size $\aleph_1$.

We recall from [J] (also see [T_3]) the definition of Juhász' Lindelöf space with points G_δ of cardinality $\beth_\omega$. $X = \bigcup_{n<\omega} X_n$, where $X_0 = \aleph_0$ and X_{n+1} is the set of non-principal ultrafilters on X_n. Given $u \in X_{n+2}$, define $u' \in X_{n+1}$ by $u' = \{A \subseteq X_n : \{v \in X_{n+1} : A \in v\} \in u\}$. Then define for any $u \in X_{n+1}, u^{(i)} \in X_{n+1}$ for $i \leq n$ by $u^{(0)} = u$, $u^{(i+1)} = u^{(i)\prime}$. A set is open if it is the union of members of X_0 and sets of form $\{u\} \cup \bigcup_{i=0}^{n} A_i$, where $A_i \in u^i$ for $0 \leq i \leq n$, and $u \in X_{n+1}$.

Theorem 21. *X is Lindelöf but has no Lindelöf subspace of cardinality $\aleph_1$.*

Proof. Suppose X has a Lindelöf subspace Y of cardinality $\aleph_1$. Then there is an $n > 0$ such that $|Y \cap X_n| = \aleph_1$. We will find a subset Z of $Y \cap X_n$ of cardinality $\aleph_1$ with no complete accumulation point. This will contradict Lindelöfness.

We first note that no point of $\bigcup_{k \leq n} X_k$ is a limit point of any subset of $Y \cap X_n$. Let $Y \cap \bigcup_{k>n} X_k = \{u_a\}_{\alpha<\omega_1}$, possibly with repetitions (if it's empty, we're done). Say $u_a \in X_{n_a}$; then $u_a^\dagger = u_a^{(n_a-(n+1))}$ is an ultrafilter on X_n. Let $\{Z_\beta\}_{\beta<\omega_2}$ be subsets of $Y \cap X_n$ of size $\aleph_1$ with pairwise intersections of size $< \aleph_1$. Claim some Z_{β_0} is the required Z. If not, then for each $\beta < \omega_2$, there is an $\alpha < \omega_1$ such that every $u \in u_\alpha^\dagger$ has intersection of size $\aleph_1$ with Z_β. But then there is an $\alpha < \omega_1$ and $\beta_1 \neq \beta_2 \in \omega_2$ such that Z_{β_1} and Z_{β_2} are in $u_\alpha^\dagger$. But then $\left(\{u_\alpha^\dagger\} \cup Z_{\beta_1}\right) \cap Z_{\beta_2}$ is uncountable, contradiction.

The following will prove useful in the discussion that follows, and is of independent interest.

Lemma 22. *Any forcing preserves the separation axioms T_i, $i \leq 3\frac{1}{2}$.*

Proof. T_0, T_1, and T_2 are immediate. For T_3, if x is in an open $U \in \mathcal{T}(G)$ (the topology generated in the extension by the topology $\mathcal{T}$ in the ground model), then there are $V, W \in \mathcal{T}$ such that $x \in V \subseteq \overline{V} \subseteq W \subseteq U$ where $\overline{V}$ is taken in $\mathcal{T}$. But the closure of V in $\mathcal{T}(G)$ is no bigger, so is also a subset of U. $T_{3\frac{1}{2}}$ is more delicate. Recall a space is completely regular (which plus $T_1 = T_{3\frac{1}{2}}$) if and only if the $f^{-1}(U)$'s, f continuous real-valued, U open in $\mathbb{R}$, form a basis for X. By definition of continuity, such $f^{-1}(U)$'s are open in $\mathcal{T}(G)$. It suffices to show that each member of $\mathcal{T}$ is of form $f^{-1}(U)$. In the ground model V this is true, but the problem is that we may have a larger real line in $V[G]$. Suppose $U = \bigcup_{a\in A} (a_\alpha, b_\alpha)^{\mathrm{V}}$. Let $U^* = \bigcup_{a\in A} (a_\alpha, b_\alpha)^{V[G]}$. Then $U^* \cap V = U$. Given $f : X \to \mathbb{R}^V$, let f^* denote the same function with range $\mathbb{R}^{V[G]}$. Then $f^{*-1}(U^*) = f^{*-1}(U)$. So each member of $\mathcal{T}$ is indeed of the appropriate form, once we verify that f^* is continuous. It suffices to show that $f^{*-1}\left((q, r)^{V[G]}\right)$ is open for rational q, r. But this equals $f^{*-1}\left((q, r)^{V}\right)$, which is open in $\mathcal{T}$.

T_4 is not in general preserved by forcing, but since Lindelöf T_3 spaces are T_4, in our situation we will get it for free. We also get $T_{3\frac{1}{2}}$ for free, but thought its preservation worth proving.

The fact that the Juhász space has cardinality $\beth_\omega$ is not totally essential:

Theorem 23. *Suppose X is a Lindelöf T_i space, $i \leq 3$, with no Lindelöf subspace of size $\aleph_1$. Then there is a forcing extension in which $|X| = \aleph_2$ and X remains a Lindelöf space with no Lindelöf subspace of size $\aleph_1$.*

Corollary 24. *It is consistent that there is a Lindelöf T_1 space of cardinality $\aleph_2 = (2^{\aleph_0})^+$ with no Lindelöf subspace of size $\aleph_1$.*

Proof of Theorem 23. We first add $\aleph_1$ Cohen reals and then collapse $|X|$ to $\aleph_2$ with conditions of size $\leq \aleph_1$. By [D_1], X remains Lindelöf.

By Lemma 22, X remains T_i. The second forcing adds no new subspaces of size $\aleph_1$; since it is countably closed, it can't make a non-Lindelöf subspace Lindelöf. Thus it suffices to show that after adding $\aleph_1$ Cohen reals, X still has no Lindelöf subspaces of size $\aleph_1$. In fact we have

Lemma 25. *Suppose X is a space in V and we force with a Property K partial order. Then if X has a Lindelöf subspace of size $\aleph_1$ in the extension, it has one in V.*

Proof. Suppose X has a Lindelöf subspace of size $\aleph_1$ in the extension. Let $1 \Vdash \dot{f} : \check{\omega}_1 \to \check{X}$, $\dot{f}$ is one-one, $\dot{f}``\check{\omega}_1$ is Lindelöf. For each $\alpha \in \omega_1$, take a maximal incompatible set of conditions $\{p_{\alpha n}\}_{n<\omega}$, and points $\{x_{\alpha n}\}_{n<\omega}$ in X such that $p_{\alpha n} \Vdash \dot{f}(\check{\alpha}) = \check{x}_{\alpha n}$. Let $S = \{x_{\alpha n} : \alpha < \omega_1, n < \omega\}$. Claim S is uncountable and Lindelöf. To see that S is uncountable, by property K take a pairwise compatible uncountable subset of $\{p_{\alpha n} : \alpha < \omega_1, n < \omega\}$ and observe that the corresponding $x_{\alpha n}$'s are distinct. To see that S is Lindelöf, suppose first that every uncountable $T \subseteq S$, $T \in V$ is forced to have uncountable intersection with $f``\omega_1$ in some extension. Then, in that extension, $T$ is forced to have a complete accumulation point in $f``\omega_1$. Thus there is a p and an x and an α such that

$$p \Vdash \text{“}\dot{f}(\check{\alpha}) = \check{x} \text{ and } \check{x} \text{ is a complete accumulation point of } \check{T}.\text{”}$$

But then x really is an accumulation point of T, and, since p is compatible with some $p_{\alpha n}$, $x = x_{\alpha n}$ is in S.

We next show that the other case can't happen: suppose there is an uncountable $T \subseteq S$, $T \in V$, such that $1 \Vdash \dot{f}``\check{\omega}_1 \cap \check{T}$ is countable. We can extend each $p_{\alpha n}$ such that $x_{\alpha n} \in T$ to a $q_{\alpha n}$ for which there is a $\beta_{\alpha n}$ such that $\forall \gamma \geq \beta_{\alpha n}$, $q_{\alpha n} \Vdash \dot{f}(\check{\gamma}) \notin \check{T}$. Then there is an uncountable $A \subseteq \omega_1 \times \omega$ such that the $\{q_{\alpha n} : \langle \alpha, n\rangle \in A\}$ are pairwise compatible. But now take some $\langle \alpha_0, n_0 \rangle \in A$ and then take $\langle \alpha_1, n_1\rangle \in A$ such that $\alpha_1 \geq \beta_{\alpha_0 n_0}$. Then $x_{\alpha_1 n_1} \in T$ and $q_{\alpha_1 n_1} \Vdash \dot{f}(\check{\alpha}_1) = \check{x}_{\alpha_1 n_1}$, but $q_{\alpha_0 n_0} \Vdash \dot{f}(\check{\alpha}_1) \notin \check{T}$, contradiction.

The second author would like to thank the members of the Toronto set theory seminar for correcting his flawed original proof of Lemma 25.

Now we move on to the more technically difficult third of the paper. The question of which forcings preserve Lindelöfness has been of interest since Shelah [S] proved the consistency from a weak compact of there being no Lindelöf space with points G_δ of size $\aleph_2$. The question of whether, in particular, countably closed forcing preserves Lindelöfness in space with points G_δ is crucial with regard to Arhangel'skiĭ's problem mentioned earlier. See [T_3]. Very recently, P. Koszmider (personal communication) has shown that it consistently need not, even for compact first countable T_2 spaces, but, by Lemma 26 below, $\sim$ CH is essential for his argument, while CH is natural to assume in the context of Arhangel'skiĭ's problem or the question of whether there exist Lindelöf first countable T_1 spaces of size $> 2^{\aleph_0}$.

As mentioned earlier, Dow [D_1] showed that Mitchell forcing preserves Lindelöfness, while Koszmider [JK] has shown that property K forcing preserves the Lindelöfness of compact spaces. It is natural to investigate the Kunen collapse K of a huge cardinal in connection with Arhangel'skiĭ's problem and the Lindelöf first countable one, and indeed we can prove that $j(K)/K$ preserves Lindelöfness, where j is the elementary embedding given by the huge cardinal, but we have been unable to use this to make any progress on these classic problems. Nonetheless, we think the preservation argument is of interest, so we present it here.

The proofs of the following assertions can be found in [K] or [Fr]. The elementary embedding $j : V \to M$ given by a large cardinal extends to a generic elementary embedding (which we also call 'j'), $j : V[G] \to M[G][H]$, where G is P-generic over V and $G * H$ is $j(P)$-generic over M, provided that P is regularly embedded in $j(P)$ and there is a *master condition* $m \in j(P)/P$ such that if H is $j(P)/P$ generic over $M[G]$ and contains m, then $p \in G$ implies $j(p) \in G * H$. If M is closed under λ-sequences and $j(P)/P$ is λ-chain condition $*$ λ-closed, then names for λ-sequences of elements of $M[G][H]$ with names in M will be in M, so $M[G][H]$ will be closed under such sequences. In particular, in the huge cardinal context and with the Kunen collapse K, one gets $M[G][H]$ is closed under sequences of size $\leq j(\kappa)$ of objects with names in M. The relevance here is that if $\mathcal{B}$ is a basis of size $\leq \aleph_2$ for a space X of size $\leq \aleph_2, j''\mathcal{B}$ and $j''X$ are both in $M[G][H]$, and the topology $j''\mathcal{B}$ generates on $j''X$ (we write '$\langle j''X, j''\mathcal{B}\rangle$') is homeomorphic to the one $\mathcal{B}$ generates on X. Since $j''A \subseteq j(A)$ for any A, $\langle j''X, j''\mathcal{B}\rangle$ is weaker than the topology $\{T \cap j''X : T \in j(\mathcal{B})\}$ generates on $j''X$. Let $\chi(x)$ be the least cardinal of a neighborhood base at x. As shown in [DTW$_1$], if $\chi(x)$ is less than the critical point of j for each $x \in X$, then the two topologies are equal. In the case of the Kunen collapse, the critical point is $\kappa = \aleph_1$, so this reduces to first countability. Note also that a first countable space has a basis of cardinality equal to the size of the space, so hugeness gets j'' of the basis into $M[G][H]$.

By elementarity, to show X has a Lindelöf subspace Y of size $\aleph_1$, it suffices to show that in $M[G][H]$, $j(X)$ has one of size $j(\aleph_1)$. $j''(X)$ with the topology generated by $j''\mathcal{B}$ is the natural candidate. Note first that $\langle X, \mathcal{B}\rangle \in M[G][H]$. $|j''X| \leq j(\kappa)$ so it's in $M[G][H]$. Similarly for $j''\mathcal{B}$. Thus $M[G][H]$ "knows" $\langle X, \mathcal{B}\rangle$ and $\langle j''X, j''\mathcal{B}\rangle$ are homeomorphic. Therefore, to prove $\langle j''X, j''\mathcal{B}\rangle$ is Lindelöf, it suffices to show that the forcing $j(P)/P$ preserves Lindelöfness over $M[G]$, for then $\langle X, \mathcal{B}\rangle$ will be Lindelöf in $M[G][H]$.

Let us now prove Theorem 5 by proving the preservation of Lindelöfness by $j(K)/K$, where K is the Kunen collapse.

We can write $K = P * \dot{Q}$, where P collapses κ to $\aleph_1$ and Q $j(\kappa)$ to $\kappa^+ = \aleph_2$; then $j(K) = j(P) * j(\dot{Q})$. K is constructed so that it regularly embeds into $j(P)$, so $j(K)/K = \big(j(P)/(P*\dot{Q})\big)*j(\dot{Q})$ is $\aleph_2$-closed and so easily preserves the Lindelöfness of a space of size $\aleph_2$ (Lemma 26 below), so the heart of the matter is to show $j(P)/(P * \dot{Q})$ (to be precise, $j(P)/(G * H)$, where $G * H$ is generic for $P * \dot{Q}$) preserves Lindelöfness, which we proceed to do via a long sequence of lemmas.

Lemma 26. *Suppose R is a λ-closed partial order, λ regular, and $\langle X, \mathcal{T}\rangle$ is a Lindelöf space, $|X| \leq \lambda$. Let G be R-generic over V. Let $\mathcal{T}(G)$ be the topology $\mathcal{T}$ generates in $V[G]$. Then $\langle X, \mathcal{T}(G)\rangle$ is Lindelöf.*

Proof. It suffices to consider open covers by members of $\mathcal{T}$. Without loss of generality, let $X = \lambda$ and, given an open cover $\mathcal{U}$ of X in $V[G]$, let $f : \lambda \to \mathcal{U}$ be such that $\alpha \in f(\alpha)$. Pick a descending sequence of conditions $\{r_\alpha\}_{\alpha<\lambda}$ such that for each α there is a $U_\alpha \in \mathcal{T}$ such that $r_\alpha \Vdash \dot{f}(\alpha) = \check{U}_\alpha$. Then $\{U_\alpha\}_{\alpha<\lambda}$ is a cover in V so has a countable subcover $\{U_\alpha\}_{\alpha\in S}$. Pick $\alpha_0 > \sup S$. Then $r_{\alpha_0} \Vdash (\{U_\alpha\}_{\alpha\in S})\check{}$ is a countable subcover of $\dot{\mathcal{U}}$.

The rest of the proof involves the interplay between elementary submodels and generic extensions; many of the lemmas are not specific to the Kunen collapse and are useful in other contexts, so we shall point out exactly what assumptions are needed. Recall that '$\prec$' is used to mean 'is an elementary submodel of '. We work in the general context of a partial order $P*\dot{Q}$, G P-generic over V, N an elementary submodel of a "sufficiently large $H(\theta)$" (see [JW, chapter 24] for elucidation of this) so that $P \in N$, $P \subseteq N$, $V_1 = V[G]$, $H_1 = H(\theta)[G]$, $N_1 = N[G] = \{\tau_G : \tau \in N$, τ

a P-name}. For Lemmas 30 and 31 we take H Q-generic over V_1, let $V_2 = V_1[H]$, $H_2 = H_1[H]$, and $N_2 = N_1[H] = \{\tau_H : \tau \in N_1, \tau$ a Q-name $\}$. We also need for these two the additional assumptions that $\kappa < \lambda$ are cardinals, that $Q \in N_1$, that Q satisfies the λ-chain condition, that $\mu = N \cap \lambda$ is a cardinal, $\kappa < \mu < \lambda$, and N is closed under sequences of length $< \mu$. The following result is known:

Lemma 27. *Suppose λ is Mahlo, $\theta \geq \lambda$ is regular, $X \in H(\theta)$, and $|X| < \lambda$. Then $\{\mu < \lambda : \mu$ is a cardinal and $(\exists N \prec H(\theta))\,(X \subseteq N, |N| < \lambda, N$ is closed under sequences of length $< \mu$, and $N \cap \lambda = \mu)\}$ is stationary.*

Proof. By Corollary 1.4 of [D$_3$], $C = \{a < \lambda : (\exists N \prec H(\theta))\,(X \subseteq N$ and $N \cap \lambda = a)\}$ includes a closed unbounded set. Since λ is Mahlo, we may pick a strongly inaccessible $\mu \in C$ with $\mu > |X|$. Let $N \prec H(\theta), X \subseteq N$, $N \cap \lambda = \mu$. Then take $N' \prec N$ such that $X \cup \mu \subseteq N'$ and $|N'| = \mu$. Then $|N'| = \mu$, $N' \cap \lambda = \mu$, $X \subseteq N'$. Now construct by induction an increasing sequence $\{N_a : a < \mu\}$, $N_\alpha \prec N'$, such that

(1) $N_\alpha \subseteq N_\beta$ for $\alpha < \beta < \mu$,
(2) $|N_\alpha| < \mu$ for all $\alpha < \mu$,
(3) $\alpha \cup [N_\alpha]^{\leq |N_\alpha|} \subseteq N_{\alpha+1}$, for all $\alpha < \mu$.

This can be done since μ is strongly inaccessible. Let $N'' = \bigcup_{\alpha<\mu} N_\alpha$. $\mu \subseteq N'' \cap \lambda \subseteq N' \cap \lambda = \mu$, so $N'' \cap \lambda = \mu$, $\mu \leq |N''| \leq |N'| = \mu$, and by construction, N'' is closed under sequences of length $< \mu$.

As discussed below, for the Kunen collapse $K = P * \dot{Q}$, we indeed have that Q satisfies the $j(\kappa)$-chain condition. By elementarity, $j(\kappa)$ is Mahlo, so Lemma 27 applies. Only in Lemmas 32 and 33 will we use specific properties of the Kunen collapse, which we shall formally define at that point.

Lemma 28. *$N_1 \prec H_1$.*

Proof. By Tarski's Criterion, it suffices to show that every existential formula with parameters in N_1 which holds in H_1 has a witness from N_1 there. So let $x_1, \cdots, x_n \in N_1$ and suppose $H_1 \models (\exists x)\varphi(x, x_1 \cdots, x_n)$. Say $x_i = (\tau_i)_G$, i.e. the G-interpretation of the P-name τ_i. Then there is a $p \in G$ such that $H(\theta) \models [p \Vdash (\exists x)\varphi(x, \tau_1, \cdots, \tau_n)]$. By the Maximum Principle, there is a name τ such that $H(\theta) \models [p \Vdash \varphi(\tau, \tau_1, \cdots, \tau_n)]$. Now $P \subseteq N$ so $p \in N$ and we can find such a $\tau \in N$. So $H_1 \models \varphi\,(\tau_G, (\tau_1)_G, \cdots, (\tau_n)_G)$ and we are done.

Lemma 29. *$N_1 \cap V = N$.*

Proof. Let $x \in N_1 \cap V$, say $x = \tau_G, \tau \in N$. Let $D = \{p \in P : p \Vdash \tau \notin \check{V}$ or $(\exists y \in \check{V}) p \Vdash \tau = \check{y}\}$. Then D is dense so there is a $p \in G \cap D$. But $p \in N$, so $N \models [(\exists y) p \Vdash \tau = \check{y}]$. Take such a y in N. Then $y = \tau_G = x$, so $x \in N$.

Lemma 30. *$N_2 \prec H_2$.*

The argument would be exactly like Lemma 28 except that we do not have $Q \subseteq N_1$ so we have to get $p \in N_1$ in another way. Suppose, as before, that $H_2 \models (\exists x)\varphi(x, (\tau_1)_H, \cdots, (\tau_n)_H)$ where $\tau_1, \cdots, \tau_n \in N_1$. $N_1 \prec H_1$ so in N_1 we can find a maximal antichain A included in Q deciding $(\exists x)\varphi(x, \tau_1, \cdots, \tau_n)$. Since $A \in N_1$, $|A| \in N_1$; since $|A| \in N_1$ and $|A| < \lambda$, $|A| < \mu$ so $A \subseteq N_1$. Let $p \in A \cap H$. $H_1 \models [p \Vdash (\exists x)\varphi(x, \tau_1, \cdots, \tau_n)]$ since p can't decide it the other way. Since $p \in N_1$, we can finish as before.

Lemma 31. *$N_2 \cap V = N$.*

Proof. This is much as in Lemma 29, but with the same trick as in Lemma 30. We show $N_2 \cap V_1 = N_1$, and then apply $N_1 \cap V = N$.

Note. Instead of V, V_1, V_2 above, it would be more correct to use $H(\theta)$, H_1, and H_2, but the notation might be a little confusing.

Let us now define the Kunen collapse and note that it in fact satisfies all the requirements for the previous lemmas.

The Kunen collapse is defined recursively as the finite support iteration of κ-chain condition partial orders. Here is one way to do it, somewhat more simply than in [K]. We follow the exposition of iterated forcing in [B]. For $\alpha < \kappa$ we determine Q_α in V^{P_α} as we define a finite support iteration $\langle P_\alpha : \alpha < \kappa \rangle$. The Silver collapse of a cardinal κ to ω_2 is a variation of the Lévy collapse; it consists of all partial functions $p : \omega_1 \times \kappa \to \kappa$ such that for all α, β, $p(\alpha, \beta) < \beta$, $|\mathrm{dom}(p)| \leq \aleph_1$, and there is a $\delta < \omega_1$ such that $\mathrm{dom}(p) \subseteq \delta \times \kappa$. We use '$V_\alpha$' for the set of all sets of rank less than α. For non-strongly inaccessible α, we take Q_α to be the trivial partial order. For strongly inaccessible α, we take Q_α to be the Silver collapse of κ to ω_2 in $V^{P_\alpha \cap V_\alpha}$. We then define $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$.

It is not difficult to verify that $P = P_\kappa$ collapses κ to ω_1. Also, κ remains a large cardinal in any $V^{P_\alpha \cap V_\alpha}$, so Q_α has the κ-chain condition in V^{P_α} and $P_{\alpha+1}$ has the κ-chain condition by induction. Let Q denote the Silver collapse of $j(\kappa)$ to ω_2 in V^P. Then $K = P * \dot{Q}$ is the Kunen collapse. Observe that $j(P)$ has the same recursive definition as P. In fact $j(P)_\kappa \cap V_\kappa = P$, and the ordering used at the next step is the Silver collapse of $j(\kappa)$ to ω_2 over V^P, namely Q. (We use here the fact that $M^{j(\kappa)} \subseteq M$, so this really is Q.) This yields a canonical embedding of $P * \dot{Q}$ into $j(P)$. For all the details of this, see [Fr].

We now note that the Kunen collapse – or, more precisely, $j(K)/K$ – satisfies the requirements of Lemmas 26-31. That $j(K)/K$ is of form '$j(\kappa)$-chain condition $*j\big(j(\kappa)\big)$-closed' is proved in [Fr]. It is easy to see by the inaccessibility of κ, the fact that P is a finite support iteration, that K is regularly embedded in $j(K)$, and the definition of the Silver collapse. Inaccessibility of κ also assures the $j(\kappa)$-chain condition for Q. By Lemma 27 we can pick a suitable μ between κ and $j(\kappa)$

and an appropriate N, so as to satisfy the conditions laid out earlier, and so that everything relevant is in N, e.g. a name $\dot{\mathcal{U}}$ for the cover in question, etc. Note that $j(P) \cap N = (j(P))_\mu \cap V_\mu$, where $(j(P))_\mu$ is defined just as P is, but with κ replaced by μ.

Lemma 32. *If J is $j(P)/(G * H)$-generic over V_2, then J is also $N_2 \cap \big(j(P)/(G * H)\big)$-generic over V_2.*

Here, $j(P)/(G * H)$ means the same thing as $j(P)/(P * \dot{Q})$, but with the generic sets specifically identified. The elements of $j(P)/(G * H)$ are all $p \in j(P)$ such that p is compatible with every element of (the subset of $j(P)$ identified with) $G * H$.

From Lemma 32, $|N_2| \leq \mu$, and the fact that μ is countable in $V^{j(P)}$, we will be able to easily show that Lindelöfness is preserved.

Proof of Lemma 32. It suffices to show that any $A \in V_2$ which is a maximal antichain in $N_2 \cap (j(P)/(G * H))$ is in fact maximal in $j(P)/(G*H)$. (Note A need not be in N_2.) Suppose $p \in j(P)/(G*H)$; we will find an $r \in A$ compatible with p. We define $p^\mu(\alpha) \in N$ recursively for $\alpha < \mu$. For $\alpha = 0$, let $p^\mu(\alpha) = p(\alpha)$. For $\alpha = \beta + 1$, given $p^\mu|\alpha \in N$, since $1 \Vdash_{j(P)_\beta} p(\alpha) \in \dot{Q}_\alpha$, $1 \Vdash_{j(P)_\beta} (\exists q)\big(q = p(\alpha)|(\mu \times \omega_1)\big)$. Therefore, by the Maximal Principle, there is a term $\dot{q}$ such that 1 and hence $p^\mu|\alpha \Vdash \dot{q} = p(\alpha)|(\mu \times \omega_1)$. Now recall $|\mathrm{dom}\big(p(\alpha)\big)| \leq \omega_1$, so that $p(\alpha)|(\mu \times \omega_1)$ is actually $p(\alpha)|(\delta \times \omega_1)$ for some $\delta < \mu$. Since $j(P)_\beta$ satisfies the μ-chain condition, and $N^{<\mu} \subseteq N$, it follows that in fact we could choose such a $\dot{q} \in N$. We will take such a $\dot{q}$ to be $p^\mu(\alpha)$. For limit α we can just use $N^\alpha \subseteq N$ to get $p^\mu(\alpha) \in N$. Having defined $p^\mu(\alpha) \in N$ for all $\alpha < \mu$, define $p^\mu = \langle p^\mu(\alpha) : \alpha < \mu \rangle$. Claim $p^\mu \in N$. Well, $N \models \Big[(\forall \alpha < \lambda) p^\mu|\alpha \Vdash p^\mu(\alpha) \in \dot{Q}_\alpha\Big]$. Therefore $N \models (\exists s \in j(P))\,(\forall \alpha < \lambda)\ s(\alpha) = p^\mu(\alpha)$. But then $s = p^\mu$ and $p^\mu \in N \cap j(P)$. $p \leq p^\mu$, so $p^\mu \in N_2 \cap (j(P)/(G * H))$. By the maximality of A, there is a $q \in A$ and an $r \in N_2 \cap (j(P)/(G * H))$ with $r \leq p^\mu, q$. We claim r and p are compatible, which will complete the proof of the lemma. Define $\bar{r}$ so that $\bar{r}(\alpha) = p(\alpha)$ for $\alpha \geq \mu$, and for $\alpha < \mu$, $\bar{r}(\alpha)$ is defined recursively so that $\bar{r}|\alpha \Vdash \bar{r}(\alpha) = r(\alpha) \cup [p(\alpha) - p(\alpha)|(\mu \times \omega_1)]$. Then we get $\bar{r}|\alpha \Vdash \bar{r}(\alpha) \leq p(\alpha)$, since $r \leq p^\mu$. It is clear that $\bar{r} \leq p$ and so we are done.

Now finally, we show $j(P)/(G * H)$ preserves Lindelöfness. The argument is the same as that used by Dow [D_1] at the end of his proof that adding at least $\aleph_1$ Cohen reals followed by countably closed forcing preserves Lindelöfness.

Let $1 \Vdash \dot{\mathcal{U}}$ is an open cover of $\check{X}$. For each $x \in X$, let $D_x = \{p \in N_2 \cap \big(j(P)/(G * H)\big)$: there is a basic open U containing x such that $p \Vdash \check{U} \in \dot{\mathcal{U}}\}$. Claim D_x is dense in $N \cap \big(j(P)/(G * H)\big)$. If so, by Lemma 32, $\{U \in N_2 : \exists q \in J \cap N_2$ such that $q \Vdash \check{U} \in \dot{\mathcal{U}}\}$, is a subcover of $\mathcal{U}$ in $V^{j(P)} = V[G][H][J]$. But $|N_2| \leq \mu$ and μ is countable in $V^{j(P)}$, so we are done.

Finally, then, to show D_x is dense, let $p \in N_2 \cap (j(P)/(G * H))$. Then $\mathcal{U}' = \{U : (\exists q \leq p)\,(q \in j(P)/(G * H)\ \&\ q \Vdash \check{U} \in \dot{\mathcal{U}})\}$ is a cover of X, so it has a countable subcover $\mathcal{V}$. We may assume $\mathcal{V}$ is in N_2 since $\mathcal{U}'$ is definable there. But then $\mathcal{V} \subseteq N_2$ so there are $U \in \mathcal{V}$ and $q \in N_2$ such that $q \Vdash \check{U} \in \dot{\mathcal{U}}$, so $q \in D_x$ and D_x is dense.

Remark. It is worth noting that a simpler version of our argument establishes that P itself or the Lévy collapse of a Mahlo or "larger" large cardinal to $\aleph_1$ via finite conditions preserves Lindelöfness, and, similarly, the Lévy collapse to $\aleph_2$ via countable conditions preserves "every open cover has a subcover of size $\leq \aleph_1$". Indeed, if κ is the large cardinal, κ-Lindelöfness transforms into Lindelöfness or $\aleph_2$-Lindelöfness respectively.

The preservation of Lindelöfness by $j(K)/K$ does not at present have any useful application in Hausdorff spaces. However, consider the weaker property of *quasi-Lindelöfness.*

Definition. A space is quasi-Lindelöf if every open cover has a countable subcollection whose closures cover.

Although quasi-Lindelöf regular spaces are Lindelöf and hence first countable quasi-Lindelöf T_3 spaces have cardinality $\leq 2^{\aleph_0}$, the same result does not hold for T_2 spaces [BY]; see also [T_4]. The following theorem, proved in the same way as for Lindelöfness, therefore is likely non-trivial even for T_2 spaces. We leave the proof to the reader.

Theorem 33. *In the Kunen model, every first countable quasi-Lindelöf space of size $\aleph_2$ has a quasi-Lindelöf subspace of size $\aleph_1$.*

In [T_4] it is shown by adding Cohen reals and collapsing as in the proof of Theorem 23 that it is consistent with CH that there is a first countable quasi-Lindelöf space of size $\aleph_2$, but it is unclear whether the method can be combined with the Kunen collapse.

The final topic we want to consider is the restriction to "$\aleph_2$" in our results. Again, we don't know whether the techniques of the proof of Theorem 23 can be combined with the Kunen collapse. Unlike the situation in [T_1] and [T_2], our preservation proofs seem to require more than just closure and chain condition considerations, so the axiomatic approach of [T_1] and [T_2] in which the existence of nice generic huge embeddings is postulated is seemingly not applicable here. One can possibly use the models of [Fo_1] or [Fo_2] to get up to $\aleph_{\mathbf{n}}$ but this does not seem worth doing until large Lindelöf first countable T_1 spaces are found.

Remark. After this work was submitted, Piotr Koszmider and the second author [KT] used countably closed forcing to construct a Lindelöf regular *P-space* (i.e. G_δ's open) of cardinality $\aleph_2 = (2^{\aleph_0})^+$ which has no Lindelöf subspace of size $\aleph_1$. They also showed it consistent – from a weak generalized Martin's Axiom plus CH plus $2^{\aleph_1} > \aleph_2$ – that there was no such space.

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