The real line in elementary submodels of set theory

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It is a truism that all of mathematics can be expressed in the language of set theory, i.e. the predicate calculus (including equality) with the single two-place relation $\in$. As with any other countable language, the language of set theory is subject to the **Löwenheim-Skolem Theorem**, which asserts that if we have some axioms – e.g. the usual axioms of set theory (ZFC) – formulated in this language and those axioms have any model at all, then they have models of all infinite sizes. Those models will have varying portions of the real line included in them; the question we are interested in here is what can we say about these subsets of $\mathbb{R}$. In particular, whether any two such subsets of the same size are isomorphic as topological spaces, linear orders, or fields.

We first need to do some preliminary work to make these questions precise. First of all there is the standard difficulty that, by Gödel's incompleteness theorem, we cannot be assured that there is any model of the axioms of set theory, and certainly not one which is a set rather than a class. The standard solution is to argue that any mathematical proof only involves finitely many axioms of set theory and one can prove within set theory the existence of nice models for these, namely $H(\theta)$, θ an uncountable cardinal, $H(\theta)$ the collection of sets which have cardinality less than θ, whose members have cardinality less than θ, whose members of members have cardinality less than θ, etc. Any particular proof in practice only uses sets up to a certain level in the set-theoretic hierarchy, e.g. reals, sets of reals, and sets of sets of reals, so one can work with $H(\theta)$, θ "sufficiently large", rather than the entire set-theoretic universe V. The non-logician reader will not lose any insight, and will indeed gain some, by replacing every occurrence of $H(\theta)$ in the rest of the paper by V.

Now given $H(\theta)$, satisfying some fragment of the axioms of set theory, by the Löwenheim-Skolem Theorem, given any non-empty subset S of $H(\theta)$ we can close off, i.e. throw in witnesses for existential quantifiers, so as to produce an elementary submodel M of $H(\theta)$ of size $|S| + \aleph_0$, i.e. a subset M of $H(\theta)$, $S \subseteq M$, such that M as a collection of sets with the membership relation that

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has the property that anything that can be said in the language of set theory about a finite sequence of elements of M is true in $H(\theta)$ if and only if it is true in M.

Given a topological space $\langle X, \mathcal{T} \rangle$, in M, we define X_M to be $X \cap M$ with topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$ [JT]. In particular, taking $H(\theta)$ such that the real line $\mathbb{R}$ is a member of $H(\theta)$, we can look at elementary submodels M of $H(\theta)$ containing $\mathbb{R}$. These are easy to obtain by closing off, starting with $\mathbb{R}$. Let us emphasize that these models contain $\mathbb{R}$ as an element but need not include it. We came to the subject of this paper from [JT] via considering $\mathbb{R}_M$ – how does it depend on M, in particular the cardinality of M? These $\mathbb{R}_M$'s are the subject of our study. In fact, though, the topological space $\mathbb{R}_M$ is nothing other than the subspace topology $\mathbb{R} \cap M$ inherits from $\mathbb{R}$, since [JT] $\mathbb{R}$ is first countable. Thus from now on we shall talk about $\mathbb{R} \cap M$ rather than $\mathbb{R}_M$. Now $\mathbb{R} \cap M$ also has an order and algebraic structure; $\mathbb{R} \cap M$ is topologically dense in $\mathbb{R}$ by elementarity, and so its linear order topology coincides with its subspace topology. Also by elementarity, it is easy to see that $\langle \mathbb{R} \cap M, +, \cdot, 0, 1 \rangle$ is a subfield of $\mathbb{R}$. The basic question we shall study in this paper is, *how many different types of $\mathbb{R} \cap M$'s can there be of the same cardinality*? The answers will vary according to the way we consider two $\mathbb{R} \cap M$'s to be the same – topologically, order-theoretically, or algebraically – and also according to what set-theoretic axioms we assume. The non-logician will be able to understand the statements of most of our results, but the proofs involve non-trivial set theory. In addition to standard forcing arguments, we will be using some basic large cardinal theory. We refer the reader to [K] for information about $0^{\#}$ and Ramsey cardinals.

Here are our principal results:

Theorem.

a) *As subfields of $\mathbb{R}$, $\mathbb{R} \cap M$ is isomorphic to $\mathbb{R} \cap N$ if and only if they are equal.*

b) *CH implies there are only two order-isomorphism (homeomorphism) types of $\mathbb{R} \cap M$'s, namely $\mathbb{Q}$ and $\mathbb{R}$.*

c) *It is consistent that there are exactly three order-isomorphism (homeomorphism) types of $\mathbb{R} \cap M$'s.*

d) *If it is consistent that there is a Ramsey cardinal, it is consistent that there are $2^{2^{\aleph_0}}$ pairwise non-homeomorphic (non-order isomorphic) $\mathbb{R} \cap M$'s of power $2^{\aleph_0}$.*

e) *It is consistent with $2^{\aleph_0} = \aleph_2$ that there are $2^{\aleph_1}$ pairwise non-homeomorphic (non-order isomorphic) $\mathbb{R} \cap M$'s of size $\aleph_1$.*

f) It is consistent that all $\mathbb{R} \cap M$'s of size $\aleph_1$ are homeomorphic but that there exist $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ of size $\aleph_1$ which are not order-isomorphic.

It turns out that the algebraic types are trivial, and that most of the results about topological types follow from those about order types. Therefore we will quickly dispose of the algebra of $\mathbb{R} \cap M$, then consider the order, and then the topology. We will end by looking at $\mathbb{R}^n \cap M$, in particular $\mathbb{C} \cap M$.

Theorem 1. *Considering $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ as subfields of $\mathbb{R}$, $\mathbb{R} \cap M$ is isomorphic to $\mathbb{R} \cap N$ if and only if $\mathbb{R} \cap M = \mathbb{R} \cap N$.*

Proof. Suppose $f : \mathbb{R} \cap M \cong \mathbb{R} \cap N$. Note that for each positive integer n,

$$f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = 1 + \cdots + 1 = n.$$

It follows that the same holds for negative integers: $-n + n = 0$ so $f(-n) + n = 0$ so $f(-n) = -n$. And for rationals: $mq = n$ implies $f(m) \cdot f(q) = f(n)$ so $m \cdot f(q) = n$ so $f(q) = q$.

Now suppose there is an $r \in \mathbb{R} \cap M$ with $r < f(r)$. Then there is a $q \in \mathbb{Q}$ $(= \mathbb{Q} \cap M)$ such that $r < q < f(r)$. Then $q - r > 0$ and is in $\mathbb{R} \cap M$ so there is a $t \in \mathbb{R} \cap M$ such that

$$t^2 = q - r.$$

So $f(q - r) = f(t^2) = (f(t))^2$. There is also a $u \in \mathbb{R} \cap N$ such that

$$u^2 = f(r) - q.$$

So $q + u^2 = f(r)$.

$$r + t^2 = q \quad \text{so} \quad f(r) + (f(t))^2 = q,$$

so $f(r) + (f(t))^2 + u^2 = f(r)$. So $(f(t))^2 + u^2 = 0$. But this is impossible.

Now we come to order. If $\mathbb{R} \cap M$ is countable, it is easy to see that by elementarity, $\mathbb{R} \cap M$ is a countable dense linear order without endpoints and so is isomorphic to $\mathbb{Q}$. Under $V = L$, there is little else to be said:

Theorem 2. *$V = L$ implies $\mathbb{R} \cap M$ is either isomorphic to $\mathbb{Q}$ or $= \mathbb{R}$.*

This follows immediately from

Lemma 3. *If $V = L$ and M is uncountable, $\mathbb{R} \subseteq M$.*

Proof. By a standard argument, there is an $\in$-isomorphism $i : M \cong$ some L_α, which is the identity on transitive subsets of M. Since M is uncountable, $\alpha \geq \omega_1$. Then $L_\alpha \supseteq \mathbb{R}$. But if $r \in L_\alpha$ and $i(x) = r$, then

$$n \in r \quad \text{if and only if} \quad n \in i(x),$$

$$\text{if and only if} \quad i(n) \in i(x),$$

since $\omega \subseteq M$ and so $i(n) = n$. But

$$i(n) \in i(x) \quad \text{if and only if} \quad n \in x,$$

so $x = r$.

Thus if $V = L$, all the $\mathbb{R} \cap M$'s of size $\aleph_1$ are trivially the same, for they all equal $\mathbb{R}$.

If $2^{\aleph_0} > \aleph_1$ it is easy to get M, N such that $|M \cap \mathbb{R}| = |N \cap \mathbb{R}| = \aleph_1$ but $M \cap \mathbb{R} \neq N \cap \mathbb{R}$, by a simple closing off argument, but getting that $M \cap \mathbb{R}$ and $N \cap \mathbb{R}$ both of cardinality $\aleph_1$, are not isomorphic is not so easy, in fact it's consistent with $2^{\aleph_0} > \aleph_1$ that it can't be done.

Theorem 4. *It is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and for every M and N such that $|\mathbb{R} \cap M| = |\mathbb{R} \cap N| = \aleph_1$, $\mathbb{R} \cap M$ is order-isomorphic to $\mathbb{R} \cap N$.*

Definition. $X \subseteq \mathbb{R}$ is $\aleph_1$-**dense** if for every $a < b \in \mathbb{R}$, $|X \cap (a, b)| = \aleph_1$.

Proof of Theorem 4. By elementarity, if $|\mathbb{R} \cap M| = \aleph_1$, $\mathbb{R} \cap M$ is $\aleph_1$-dense. Baumgartner [B$_1$] forced to obtain a model in which every two $\aleph_1$-dense sets of reals are order-isomorphic and (hence) $2^{\aleph_0} > \aleph_1$, establishing Theorem 4. (Actually, he proved this with defining $\aleph_1$-dense as "between any two elements, there are $\aleph_1$-elements", but saying "all $\aleph_1$-dense sets are isomorphic" is true for the one definition if and only if for the other [SW]. We can vary Theorem 4 as follows:

Theorem 5. *It is consistent with ZFC that for every M and N of size $\aleph_1$, $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ are order-isomorphic.*

Proof. It suffices to note

Lemma 6. *Suppose $|\omega_1 \cap M| = \aleph_1 = |\mathbb{R} \cap M|$. Then $\mathbb{R} \cap M$ is $\aleph_1$-dense.*

Proof. By elementarity, for every $a, b \in \mathbb{R} \cap M$, if $a < b$ there is an injection $f : \omega_1 \cap M \to (a, b) \cap M$.

$0^{\#}$ is a complicated set of natural numbers, the existence of which has large cardinal strength. Its existence is equivalent to the existence of an elementary embedding $j : L_\alpha \to L_\beta$ for some α and β, such that some ordinal less than $|\alpha|$ is moved (Kunen, see e.g. [K, p.277]), and also to the failure of *Jensen's Covering Lemma for L.*

Lemma 7. *If $0^{\#}$ does not exist, then if $|M| \geq \kappa$, then $M \supseteq \kappa$.*

Proof. Consider the Mostowki isomorphism, $i : M \cong T$, T transitive. Since $|M| \geq \kappa$, $T \supseteq \kappa$ and hence L_κ. Then $i^{-1} : T \cong M \subseteq H(\theta)$ is an elementary embedding and therefore so is $i^{-1}|L_\kappa : L_\kappa \to L_\theta$. But if $M \not\supseteq \kappa$, some $\alpha < \kappa$ gets moved.

Thus, to prove Theorem 5, start with L and perform Baumgartner's forcing to get all $\aleph_1$-dense sets of reals are order-isomorphic. $0^{\#}$ cannot be added by set forcing over L (see e.g. [K, p.186]), so in the resulting model, any uncountable M will include ω_1. $|\omega_1 \cap M|$ is always $\leq |M \cap \mathbb{R}|$, so if $|M| = \aleph_1$, we conclude $M \cap \mathbb{R}$ is $\aleph_1$-dense.

Lemma 7 gives us

Theorem 8. *If $0^{\#}$ does not exist and $|\mathbb{R} \cap M| = 2^{\aleph_0}$, then $\mathbb{R} \subseteq M$.*

Corollary 9. *In the model of Theorem 5, there are only 3 isomorphism types of $\mathbb{R} \cap M$'s.*

Proof. By elementarity and by taking θ sufficiently large, we may assume there is in M a bijection between the cardinal $2^{\aleph_0}$ and $\mathbb{R}$. But by Lemma 7, $2^{\aleph_0} \subseteq M$, so therefore $\mathbb{R}$ is. the corollary immediately follows.

It follows from Theorem 8 that if CH and $0^{\#}$ does not exist, then $\mathbb{R} \cap M$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{R}$. Surprisingly, I. Farah has improved this to get the following result, which we include with his kind permission.

Theorem 10. *CH implies that if $\mathbb{R} \cap M$ is uncountable, then $\mathbb{R} \cap M = \mathbb{R}$ and hence there are only 2 isomorphisms types of $\mathbb{R} \cap M$'s.*

Proof. By CH, there is a bijection $f : \omega_1 \to \mathbb{R}$. Hence there is such a bijection $f \in M$ and $f''(\omega_1 \cap M) = \mathbb{R} \cap M$. $\mathbb{R} \cap M$ is uncountable, so $\omega_1 \cap M$ is uncountable and hence $= \omega_1$. But then $\mathbb{R} = f''\omega_1 = \mathbb{R} \cap M$.

This is quite a contrast to the situation under CH for order types of subsets of $\mathbb{R}$ which need not be of form $\mathbb{R} \cap M$- there are $2^{2^{\aleph_0}}$ different ones - see e.g. [B$_2$].

Remark. Notice the difference between Lemma 3 and Theorem 10: in the former, we assume M is uncountable, in the latter $\mathbb{R} \cap M$ is uncountable. By Lemma 7, the nonexistence of $0^{\#}$ makes up the difference: if M is uncountable, it includes ω_1; there is an injection from ω_1 into $\mathbb{R}$; the range of the injection in M is uncountable. However, assuming CH plus *Chang's Conjeture*, there are indeed uncountable models M for which $\mathbb{R} \cap M$ is countable.

We can get sharper, axiomatic versions of Theorems 4 and 5 and in the process obtain a theorem distinguishing consistently between the number of order types of $\aleph_1$-dense sets and the number of order-types of $\mathbb{R} \cap M$'s of size $\aleph_1$. First, some definitions from [ARS].

Definition. Let X be a second countable space of size $\aleph_1$. Let $D(X) = X \times X - \{\langle x, x\rangle : x \in X\}$. An open cover $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ of $D(X)$ consisting of symmetric sets is called an **open coloring** of X. $A \subseteq X$ is $\mathcal{U}$**-homogeneous** if for some $i < n$, $D(A) \subseteq U_i$.

[ARS] call the assertion that for every such X and $\mathcal{U}, X$ can be partitioned into countably many $\mathcal{U}$-homogeneous sets "OCA"; the second author is grateful to S. Todorcevic for informing him that this is not equivalent to what is now known as "OCA" (see [T]) and that the proof in [ARS] that MA plus their OCA (which we will call "OCA_1") implies $2^{\aleph_0} = \aleph_2$ is incorrect, although the conjunction of these three hypotheses is consistent.

Definition. A set $A \subseteq \mathbb{R}$ of cardinality $\aleph_1$ is called an **increasing set** if for every $n \in \omega$ and any set $\{\langle a(\alpha, 0), \ldots, a(\alpha, n-1)\rangle : \alpha < \omega_1\} \subseteq A^n$ of pairwise disjoint n-tuples there are $\alpha, \beta < \omega_1$, such that for every $i < n$, $a(\alpha, i) < a(\beta, i)$. **ISA** is the assertion that an increasing set exists.

Theorem 11. *MA plus OCA_1 implies if $|\mathbb{R} \cap M| = |\mathbb{R} \cap N| = \aleph_1$, then $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ are isomorphic.*

Corollary 12. *MA plus OCA_1 plus $2^{\aleph_0} = \aleph_2$ plus $0^{\#}$ doesn't exist implies there are exactly 3 order-types of $\mathbb{R} \cap M$'s.*

In order to prove these, we first need three results from [ARS].

Lemma 13. *Assume MA plus OCA_1 plus not ISA. Then every two $\aleph_1$-dense sets of reals are isomorphic.*

Lemma 14. *Assume MA plus OCA_1 plus ISA. Then there is an increasing $\aleph_1$-dense set A such that $A, A^*(= \{-a : a \in A\})$ and $A \cup A^*$ are homogeneous, and every homogeneous $\aleph_1$-dense set is isomorphic to one of these three sets.*

Lemma 15. *If A is increasing, $A \ncong A^*$.*

Proof of Theorem 11. By elementarity, any $\mathbb{R} \cap M$ is homogeneous, and $(\mathbb{R} \cap M)^* = \mathbb{R} \cap M$. Therefore, assuming ISA, every $\mathbb{R} \cap M$ of size $\aleph_1$ is isomorphic to $A \cup A^*$. On the other hand, by Lemma 14, we also have only one type of $R \cap M$ of size $\aleph_1$ if ISA fails.

Corollary 13 follows as usual. The conjunction of MA, OCA_1, and ISA is interesting in that it gives an example to show the structure of $\mathbb{R} \cap M$-types may differ from that of the $\aleph_1$-dense types, even without CH.

So far, we have been looking at models in which the number of isomorphism types of $\mathbb{R} \cap M$'s has been very small; however it is indeed consistent that it be as large as possible, i.e. $2^{2^{\aleph_0}}$, assuming large cardinals, in particular, a Ramsey cardinal.

Definition. κ is Ramsey if and only if $\kappa \to (\kappa)_2^{<\omega}$.

Comparing with our previous assumption, let us note that the existence of a Ramsey cardinal implies $0^\#$ exists, but not vice versa.

Theorem 16. *If it is consistent that there is a Ramsey cardinal, it is consistent that there are $2^{2^{\aleph_0}}$ pairwise non-homeomorphic and hence non-isomorphic $\mathbb{R} \cap M$'s of size $2^{\aleph_0}$.*

Proof. We will use the Ramsey cardinal to obtain a model in which there are $2^{2^{\aleph_0}}$ distinct $\mathbb{R} \cap M$'s; on the other hand, by Lavrentieff's Theorem (see e.g. [E, 4.3.21]) any homeomorphism from $S \subseteq \mathbb{R}$ to $T \subseteq \mathbb{R}$ can be extended to a homeomorphism from S' to T', where S' and T' are G_δ subsets of $\mathbb{R}$. There are only $2^{\aleph_0}$ many such homeomorphisms, so each $\mathbb{R} \cap M$ is homeomorphic to at most $2^{\aleph_0}$ many $\mathbb{R} \cap N$'s, so if there are $2^{2^{\aleph_0}}$ distinct ones, there are $2^{2^{\aleph_0}}$ pairwise non-homeomorphic ones.

Let κ be Ramsey. Let P be the partial order for adding κ Cohen reals. The idea of the proof is that the Ramsey cardinal gives us a set I of κ indiscernibles. The submodels M of the generic extension determined by independent subsets of I will yield distinct $\mathbb{R} \cap M$'s. Now for the details.

Definition. For a model M and $I \subseteq \kappa \cap M$, κ an ordinal, I is a **set of indiscernibles** for M if and only if for every formula $\varphi(v_1, \ldots, v_n)$ and $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ all in I, $M \models \varphi[x_1, \ldots, x_n]$ if and only if $M \models \varphi[y_1, \ldots, y_n]$.

Lemma 17. *(Silver, see e.g. [K, p.100]). κ Ramsey implies there is a set of indiscernibles $I \in [k]^\kappa$ for any $M \supseteq \kappa$.*

Let $I \subseteq \kappa$, $|I| = \kappa$ be a set of indiscernibles for $H(\theta) \supseteq \kappa$, θ sufficiently large. Let $\{I_\alpha\}_{\alpha<2^\kappa}$ be an independent family of subsets of I. Let $\mathcal{H}(I_\alpha) \cap \{\kappa, I\}$ be the Skolem hull of I_α in $H(\theta)$. Since no indiscernible can be defined from the others, $\mathcal{H}(I_\alpha) \cap I = I_\alpha$. Let G be P-generic over V and hence $H(\theta)$. In $V[G]$ take $M_\alpha = \{\tau_G : \tau \text{ is a } P\text{-name and } \tau \in \mathcal{H}(I_\alpha)\}$.

By a standard argument [S. 3.2.1] M_α is an elementary submodel of $H(\theta)^{V[G]}$. We now claim that $\alpha \neq \beta$ implies (in $V[G]$) that $\mathbb{R} \cap M_\alpha \neq \mathbb{R} \cap M_\beta$. Another standard argument [S, 3.2.13] shows $M_\alpha \cap I = \mathcal{H}(I_\alpha) \cap I = I_\alpha$. In $V[G]$ there is a bijection $f : \kappa \to \mathbb{R}$. We may assume there is a P-name τ for f which is in every $\mathcal{H}(I_\alpha)$. Then f is in M_α and in M_β. Take $i \in I_\alpha - I_\beta$. Then $f(i) \in \mathbb{R} \cap M_\alpha$, but $f(i) \notin \mathbb{R} \cap M_\beta$, else $i \in \mathcal{H}(I_\beta)$, which is impossible.

Without the large cardinal, we can still get a large number of types, but of size $\aleph_1$ rather than $2^{\aleph_0}$:

Theorem 18. *Adjoin $\aleph_2$ Cohen reals to a model of GCH. Then there exist $2^{\aleph_1}$ non-homeomorphic (and hence non-isomorphic) $\mathbb{R} \cap M$'s of size $\aleph_1$.*

Note that we can have $0^\#$ not existing in such a model, so the number of types of $\mathbb{R} \cap M$'s of a certain size bears no linear relationship to that size.

To prove Theorem 18, let $\{M_\alpha\}_{\alpha<\omega_2}$ be an increasing sequence of countably closed elementary submodels of $H(\theta)$, each of size $\aleph_1$, with $\omega_2 \subseteq \bigcup_{\alpha<\omega_2} M_\alpha$. By taking a subsequence, without loss of generality we may assume that $\alpha < \beta$ implies $|\omega_2 \cap (M_\beta - M_\alpha)| = \aleph_1$. Let G be $Fn(\aleph_2, 2)$-generic over V. Let $M'_\alpha = M_\alpha[G|M_\alpha]$, where $G|M_\alpha = \{p \in G : \; dom\, p \subseteq M_\alpha\}$. Then, by a standard argument [S, 3.2.12], M'_α is an elementary submodel of $H(\theta)^{V[G]}$.

Note that since M_α is countably closed, $\mathbb{R} \cap M'_\alpha = \mathbb{R} \cap V[G|A_\alpha]$, which we will call '$\mathbb{R}_\alpha$' for short. If two $\mathbb{R}_\alpha$'s were homeomorphic, by Lavrentieff's Theorem we could assume the map is a restriction of a homeomorphism between two G_δ sets of reals and hence is coded by a real. Suppose $\dot{h}$ is a name for a function h in $V[G]$ such that for some $\alpha < \beta$, $h|\mathbb{R}_\alpha$ is a homeomorphism from $\mathbb{R}_\alpha$ to $\mathbb{R}_\beta$. We may assume $\dot{h}$ has countable support. Take any real $s \notin V$ with support of $\dot{s}$ included in $A_\beta - (A_\alpha \cup \text{ support } \dot{h})$. Then we claim

$$1 \Vdash \dot{s} \in \mathbb{R}_\beta - \dot{h}``\mathbb{R}_\alpha.$$

That $1|\vdash \dot{s} \in \dot{\mathbb{R}}_\beta$ is clear. Suppose that there were a condition p and a name $\dot{r}$ such that

$$p|\vdash \dot{r} \in \dot{\mathbb{R}}_\alpha \quad \text{and} \quad \dot{s} \;=\; \dot{h}(\dot{r}).$$

Let $\dot{r}_1 = \dot{r}|M_\alpha = \{<p, \check{n}>: <p, \check{n}> \in \dot{r} \text{ and } dom p \subseteq M_\alpha\}$. Then $(\dot{r})_G = (\dot{r})_{G|M_\alpha} = (\dot{r}_1)_G$. So $r = (\dot{r})_G$ is in $V[G|(M_\alpha \cup \text{support } h)]$ as is h, but not s. But then we can extend p outside of $M_\alpha \cup$ support $\dot{h}$ so as to ensure s is not equal to $h(r)$.

Now let us turn to homeomorphism types of $\mathbb{R} \cap M$'s. Our theorems restricting the number of isomorphism types apply in this context, while our examples of models with large number of types were in fact for homeomorphism types. One might conjecture that $\mathbb{R} \cap M$'s are homeomorphic if and only they are isomorphic, but this is consistently not the case:

Theorem 19. *It is consistent that every two $\mathbb{R} \cap M$'s of cardinality $\aleph_1$ are homeomorphic, but there exist $2^{\aleph_1}$ many $\mathbb{R} \cap M$'s of cardinality $\aleph_1$ that are pairwise non-order-isomorphic.*

Proof. First of all, we need the following:

Theorem 20. *Given any two $\aleph_1$-dense sets of reals, there is a σ-centered P which forces them to be homeomorphic.*

Corollary 21. *MA(σ-centered) plus $2^{\aleph_0} > \aleph_1$ implies any two $\aleph_1$-dense sets of reals are homeomorphic.*

This is apparently folklore. A closely related result with essentially the same proof appears in [BB]. Repeating that proof for the reader's convenience, here is a proof for Theorem 20. In fact it works for "κ-dense".

Proof. First note that adding one Cohen real makes any ground model set of reals 0-dimensional, since it creates a dense set of reals not in the ground model.

Let A, B be $\aleph_1$-dense. We first force with $Fn(\omega, 2)$. Let $\mathcal{A} = \{A_n\}_{n<\omega}$ and $\mathcal{B} = \{B_n\}_{n<\omega}$ be countable clopen bases for A, B respectively, after we have added the Cohen real. Let P be the set of all quadruples $\langle g, h, J, K\rangle$ such that

1) g is a finite one-to-one function with dom $g \subseteq A$ and ran $g \subseteq B$.
2) J is a finite collection of disjoint elements of $\mathcal{A}$.
3) K is a finite collection of disjoint elements of $\mathcal{B}$.
4) h is a one-to-one function from J onto K.
5) for every $x \in$ dom g and every $Y \in J$, $x \in Y \iff g(x) \in h(Y)$.

Order P by putting $\langle g, h, J, K\rangle \leq \langle g', h', J', K'\rangle \iff$

a) $g' \subseteq g$,

b) For every $Y \in J$, there is a $Y' \in J$ such that $Y \subseteq Y'$ and $h(Y) \subseteq h'(Y')$.

To see that P is σ-centered, first let Q be the partial order of all finite, partial one-one functions from A to B. Then Q is isomorphic to a dense subset of the partial order for adding $\aleph_1$ Cohen reals. Therefore Q is σ-centered. Since for each $g \in Q$ there are only countably many $\langle h, J, K\rangle$ such that $\langle g, h, J, K\rangle \in P$, it is easy to see that P is σ-centered. But then $Fn(\omega, 2) * P$ is σ-centered.

For each $n \in \omega$, let $D_0(n)$ be the set of all $\langle g, h, J, K\rangle \in P$ such that every element of $J \cup K$ has diameter less than 2^{-n}. For each $a \in A$ let $D_1(a)$ be the set of all $\langle g, h, J, K\rangle \in P$ such that $a \in \text{dom } g$. For each $b \in B$, let $D_2(b)$ be the set of all $\langle g, h, J, K\rangle \in P$ such that $b \in \text{ran } g$. Clearly all of these sets are dense in P so by MA(σ-centered) there is a filter G on P which meets all of them. Then $f = \bigcup\{g : \exists h, J, K\langle g, h, J, K\rangle \in G\}$ is a homeomorphism from A onto B.

Now we can proceed to prove Theorem 19. We start with a model V of GCH and perform a finite support iteration, $\{P_\alpha\}_{\alpha<\omega_2}$. Letting R_α be a P_α-name for the reals in V^{P_α}, $P_{\alpha+1}$ is defined to be $Fn(\omega_1 \times \omega, 2) * \dot{Q}_\alpha$, where $1 \Vdash_{Q_\alpha}$ all $\dot{\mathbb{R}}_\xi$, $\xi \le \alpha$, are homeomorphic. Since a finite support iteration of $\le \aleph_1$ σ-centered partial orders is σ-centered, $Q_{\alpha+1}$ is σ-centered over $V^{P_\alpha * Fn(\omega_1\times w,2)}$.

In V, take countably closed elementary submodels M and N of $H(\theta)$ such that $|M| = |N| = \aleph_1$, $M \cap \omega_2 = \alpha \in \omega_2$, $N \cap \omega_2 = \beta \in \omega_2$, $cf(\alpha) = cf(\beta) = \omega_1$, and $\alpha < \beta$. By standard arguments, in $V^{P_{\omega_2}}$, M^{P_α} and N^{P_β} are elementary submodels of $H(\theta)^{P_{\omega_2}}$. Consider $\mathbb{R}_\alpha = \mathbb{R} \cap M^{P_\alpha} = \mathbb{R} \cap V^{P_\alpha}$, $\mathbb{R}_\beta = \mathbb{R} \cap N^{P_\beta} = \mathbb{R} \cap V^{P_\beta}$. By construction, $\mathbb{R}_\alpha$ is homeomorphic to $\mathbb{R}_\beta$, but we claim they are not isomorphic. We can get ω_2 such pairs α, β so this will prove the theorem. The key observation is that, considering $V^{P_\beta} = V^{P_\alpha * F_n(\omega_1\times\omega,2) * \dot{Q}}$ where Q is σ-centered, any isomorphism from $\mathbb{R}_\beta$ to $\mathbb{R}_\alpha$ would have to take those added $\aleph_1$ Cohen reals and send them back into $\mathbb{R}_\alpha$. We will show that can't be done. First of all, it is an easy exercise to prove that if a σ-centered (in fact, property (K)) partial order forces an order-preserving injection from an uncountable ordered set $\langle A_1, \le_1\rangle$ into $\langle A_2, \le_2\rangle$, then there is an uncountable subset X of A_1 in the ground model and a ground model order-preserving injection from X into A_2. But we will show

Lemma 22. *Add $\aleph_1$ Cohen reals over any V. Then there is no order-preserving injection from an uncountable subset of the Cohen reals into $\mathbb{R}^V$.*

Proof. If there were, as in the proof of Theorem 18, g would extend to a map we could code by a real. Without loss of generality, since that code depends on only countably many Cohen reals, we may as well assume it is in V. Enumerate the Cohen reals as $\{c_\alpha\}_{\alpha<\omega_1}$. We can think of the uncountable set of Cohen reals as given by $\{c_{f(\alpha)} : \alpha < \omega_1\}$, where $p \Vdash \dot{f} : \check{\omega}_1 \xrightarrow{1-1} \check{\omega}_1$. Take $\{p_\gamma\}_{\gamma<\omega_1}, \{\xi_\gamma\}_{\gamma<\omega_1}, \{r_\gamma\}_{\gamma<\omega_1}$, $p_\gamma \in Fn(\omega_1 \times \omega, 2)$, $\xi_\gamma \in \omega_1$, $r_\gamma \in \mathbb{R}$, such that $p_\gamma \le p$ and

$p_\gamma \Vdash \dot{f}(\check{\gamma}) = \check{\xi}_\gamma$ & $\check{g}(\dot{c}_{\xi_\gamma}) = \check{r}_\gamma$. (We really should say the function coded by the code of g, rather than "$\check{g}$".)

Now take an uncountable $S \subseteq \omega_1$ such that $\{p_\gamma\}_{\gamma \in S}$ are all compatible. Without loss of generality, the supports of the p_γ's form a Δ-system with root r. Take $\gamma \neq \delta \in S$ such that $r_\gamma < r_\delta$ and $\xi_\gamma \notin$ support p_δ land $\xi_\delta \notin$ support p_γ. We can then extend $p_\gamma \cup p_\delta$ to a q with $\xi_\gamma \in$ dom q and $q \Vdash \dot{c}_{\xi_\gamma} > \dot{c}_{\xi_\delta}$, contradiction.

Now let us turn to two dimensions and consider $\mathbb{C} \cap M$, where $\mathbb{C}$ is the set of complex numbers. Again, we first turn to algebra and consider $\mathbb{C} \cap M$ as a subfield of $\mathbb{C}$.

Theorem 23. *CH is equivalent to the assertion that if $\mathbb{R} \cap M$ is uncountable, $\langle \mathbb{C} \cap M, +, \cdot, 0, 1 \rangle \cong \langle \mathbb{C} \cap N, +, \cdot, 0, 1 \rangle$ implies $\langle \mathbb{R} \cap M, +, \cdot, 0, 1 \rangle \cong \langle \mathbb{R} \cap N, +, \cdot 0, 1 \rangle$.*

Proof. We have seen that the conclusion merely says $\mathbb{R} \cap M = \mathbb{R} \cap N$; on the other hand, if $\mathbb{R} \cap M$ and hence $\mathbb{C} \cap M$ is uncountable, the hypothesis is equivalent to $|\mathbb{R} \cap M| = |\mathbb{R} \cap N|$. To see this, note that $\mathbb{C} \cap M$ is, by elementarity, an algebraically closed field of characteristic 0. Any two such fields of the same uncountable cardinality are isomorphic. Under CH, $\mathbb{R} \cap M$ uncountable implies $\mathbb{R} \cap M = \mathbb{R}$ and the implication in the theorem is trivially true. On the other hand, if $2^{\aleph_0} > \aleph_1$ and we construct distinct $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ of cardinality $\aleph_1$, they will not be field-isomorphic but $\mathbb{C} \cap M$ and $\mathbb{C} \cap N$ will be.

For the countable case, any two countable $\mathbb{C} \cap M$'s are also isomorphic, since each has a countable transcendence base over the set of algebraic numbers, which is included in M by elementarity. On the other hand, we can construct distinct countable $\mathbb{R} \cap M$'s.

It is interesting to note that although $\mathbb{R}$ and $\mathbb{C}$ are not homeomorphic, $\mathbb{R} \cap M$ and $\mathbb{C} \cap M$ may be:

Theorem 24. *ZFC does not decide whether $\mathbb{R} \cap M$ and $\mathbb{C} \cap M$ are homeomorphic.*

Proof. If $\mathbb{R} \cap M$ and (hence) $\mathbb{C} \cap M$ are countable, they are homeomorphic in ZFC, since as countable metrizable spaces without isolated points, they are both homeomorphic to $\mathbb{Q}$ [Si]. If CH and $|\mathbb{R} \cap M|$ is uncountable, then $|\mathbb{R} \cap M| = \mathbb{R}$ and $|\mathbb{C} \cap M| = \mathbb{C}$, so they are of course not homeomorphic.

Theorem 25. *Assume MA(σ-centered) plus $2^{\aleph_0} > \aleph_1$. Then if P and Q are Polish (i.e. separable completely metrizable) spaces without isolated points in M and N respectively then if $|P \cap M| = |Q \cap N| < 2^{\aleph_0}$, then $P \cap M$ is homeomorphic to $Q \cap N$.*

Corollary 26. *Assume MA(σ-centered) plus $2^{\aleph_0} = \aleph_2$ plus $0^\#$ does not exist. Then $\mathbb{R} \cap M$ and $\mathbb{C} \cap M$ are homeomorphic if they have the same cardinality less than $2^{\aleph_0}$ but not if they have cardinality $2^{\aleph_0}$.*

The corollary follows since $\mathbb{C}$ and $\mathbb{R}$ are Polish but $\mathbb{C}$ and $\mathbb{R}$ are not homeomorphic, yet if $|\mathbb{R} \cap M| = 2^{\aleph_0}$ then both $\mathbb{R}$ and $\mathbb{C}$ are included in M.

To prove the Theorem, note that since $|P \cap M| = |Q \cap N| < 2^{\aleph_0}$, they both can be embedded in the Cantor set $\mathbb{K}$, since they both are 0-dimensional spaces with a countable base. The closures of their images are both homeomorphic to the Cantor set, because $P \cap M$ and $Q \cap N$ have no isolated points and so the closures of their images are compact 0-dimensional metric spaces without isolated points. By mapping one Cantor set to another, we may then assume that both $P \cap M$ and $\mathbb{Q} \cap N$ are dense subspaces of K.

The countable case is clear, so assume $|P \cap M| = |Q \cap N| = \aleph_1$. By complete metrizability, separability, and MA(σ-centered), every open subset of P or Q has cardinality $2^{\aleph_0}$. Since any open subset of $\mathbb{K}$ intersects $P \cap M$ and $Q \cap N$ in open subsets of those spaces, we have that $|P \cap M \cap U| = |Q \cap N \cap U| = |P \cap M| = |Q \cap N|$, for any U open in K. But then by the proof of Corollary 22, $P \cap M$ and $Q \cap N$ are homeomorphic.

Getting back to $\mathbb{R}$, a stronger assertion than that there is a homeomorphism between two κ-dense sets A, B is the assertion that there is an autohomeomorphism h of R such that $h``A = B$. Steprāns and Watson [SW] show that this is equivalent to any such sets being order-isomorphic, and so for say $\kappa = \aleph_1$, requires more than MA. On the other hand, if one works with $\mathbb{K}$ rather than $\mathbb{R}$, one only needs MA(σ-centered): the homeomorphism of Corollary 22 extends [BB]. Surprisingly, this also works for κ-dense subsets of $\mathbb{R}^n, n > 1$ [SW].

We close with a question. It is a long-standing open problem whether it is consistent with $2^{\aleph_0} > \aleph_2$ that all $\aleph_2$-dense sets of reals are order-isomorphic; a less demanding question is: Is it consistent with $2^{\aleph_0} > \aleph_2$ that all the $\mathbb{R} \cap M$'s of size $\aleph_2$ are order-isomorphic?

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