Homework 1

Expanders and Pseudorandom Graphs (Fall 2024)
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Due Date: October 28.

Read and understand the statements of all questions. Answer any 2 questions.

1. Let $\mathbb{F}_2$ be the finite field with 2 elements.

   Define the graph G with vertex set $V = \mathbb{F}_2^t \setminus \{0\}$ and edge set $E = \{\{x, y\} \mid \langle x, y\rangle = 0\}$. (The inner produce $\langle \cdot, \cdot \rangle$ is over $\mathbb{F}_2$). Let $n = |V|$.

   Show that any vertex x has $(1/2 + o(1))n$ neighbors. Show that any two distinct vertices x, y have $(1/4 + o(1))n$ common neighbors.

   Use this to conclude that the number of C_4 in G is $(1/16 + o(1))n^4$, and that G is $o(1)$-C_4-quasirandom (and hence also $o(1)$-eigenvalue-quasirandom and $o(1)$-discrepancy-quasirandom).

2. Let G be a d-regular n-vertex graph and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix.

   (a) Show that G is bipartite if and only if $\lambda_n = -d$.

   (b) Show that the size any independent set in G is at most:

   $$\frac{-\lambda_n}{d - \lambda_n} \cdot n.$$

   This is called the Hoffman bound.

   (c) What can you conclude about the number of colors needed to color a d-regular (ϵd)-absolute-eigenvalue-expander graph?

3. Let G be a connected d-regular n-vertex graph. Let λ_2 be the second largest eigenvalue of the adjacency matrix A of G. Let $f : V_G \to \mathbb{R}$ be a function with $\sum_v f(v) = 0$ and $\sum_v f(v)^2 = 1$. (Thus f is spanned by all the eigenvectors of $\lambda_2, \ldots, \lambda_n$).

   (a) Show that $\max_{u,v \in V_G}(f(u) - f(v)) \geq \frac{C}{\sqrt{n}}$, for some universal constant C.

   (b) Let u and v be vertices witnessing the above inequality. Suppose ℓ is the distance between u and v. Show that

   $$\langle f, Lf \rangle \geq \Omega(\frac{1}{n\ell}),$$

   where L is the matrix $I - A$.

   (c) Thus show that in any connected d-regular graph $\lambda_2 \leq d - \frac{1}{n^2}$.

(d) Define a lazy random walk on a graph to be the following process. v_0 is a fixed vertex v^*; for each $k \geq 0$, v_{k+1} is chosen as follows: with probability 1/2, $v_{k+1} = v_k$, and with probability 1/2, v_{k+1} is a uniformly random neighbor of v_k.

Use the bound on λ_2 to show that for any vertex w in the connected component of v^*, the lazy random walk visits w within $O(n^2 \log n)$ steps with probability $1 - o(1)$.

(e) (Not to be turned in): On a cycle of length n, how many steps of the lazy random walk starting at some vertex do we have to take to have a $1 - o(1)$ probability of reaching the diametrically opposite point?

4. Let $p \in (0, 1)$ be a constant. Use the Chernoff bound and union bound to show that for $G = (V, E)$ coming from $G(n, p)$,

$$\Pr\left[\exists S, T \subseteq V \text{ such that } |e(S,T) - p|S||T|| > \epsilon n^2\right] < \exp(-\epsilon^{10} n^2).$$

5. Look up the Catalan numbers C_k on Wikipedia.

Show that for any d-regular graph G, if A is the adjacency matrix of G, then

$$\mathsf{Tr}(A^{2k}) \geq (d-1)^k \cdot C_k.$$

Conclude that G must have some eigenvalue other than its top eigenvalue have absolute value at least $2\sqrt{d-1} - o(1)$.