More Miscellaneous Problems

1. Prove that if a, b are relatively prime positive integers there do not exist positive integers m, n such that $ma + nb = ab - a - b$.

2. Let $\varepsilon \in (0,1)$ be fixed and define the function $f(x,y) = x^\varepsilon - y^\varepsilon$. Prove that for every real number r there exists two sequences of positive integers $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ such that $\lim_{i \longrightarrow \infty} f(n_i, m_i) = r$.

3. Let $t(a_1, \ldots, a_5)$ be the number of nonzero coefficients of the polynomial $(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)$. Determine, with proof, the minimum of $t(a_1, \ldots, a_5)$ over all choices of distinct real numbers a_1, a_2, a_3, a_4, a_5.

4. Suppose n is a positive integer and P is a set of $2n$ distinct points in the plane. Prove that it is always possible to pair up the points of P so that the segments joining each pair do not intersect each other.

5. There is a room that is 10 feet by 10 feet by 20 feet. A beetle is along one of the 10 by 10 foot walls, 5 feet in from either side and 1 foot up from the floor. The beetle walks to a spot on the opposite wall 5 feet in from either side and 1 foot down from the ceiling. If the beetle follows the shortest path, how long does he walk?

6. If n is a nonnegative integer and x is any real number we define $\binom{x}{n}$ to be the polynomial $x(x-1)\ldots(x-(n-1))/n!$. (Observe that if x is a positive integer then this agrees with the usual definition.) Say that a polynomial $P(x)$ is integer valued if $P(k)$ is an integer for any integer k. Prove that a polynomial $P(x)$ is integer valued if and only if there is a nonnegative integer n and integers $a_0, a_1, \ldots, a_n$ so that $P(x) = \sum a_i \binom{x}{i}$.

7. Define the infinite sequence S^1 to have terms $1, 2, 3, 4, 5, \ldots$ and for $j \geq 1$ define the sequence S^j in terms of the sequence S^{j-1}: S^j is obtained from S^{j-1} by adding 1 to every term that is a multiple of $j - 1$. For example S^2 is $2, 3, 4, 5, 6, \ldots$ and S^3 is $3, 3, 5, 5, 7, 7, 9, 9, \ldots$. Say that an integer $j \geq 2$ is special if the first $j - 1$ terms of S^j are equal to j. Determine (with proof) a simple rule for telling which integers are special.

8. Suppose that $F(x)$ is a function from the real numbers to the real numbers that satisfies $F(x)F(y) - F(xy) = x + y$ for all real x, y. What is $F(x)$?

9. Determine all 4-tuples of real numbers a, b, c, d such that $a+b+c = d$ and $1/a+1/b+1/c = 1/d$.