MATH 642:491–Fall, 2013

Problems on 2-way counting and inclusion-exclusion

1. A frog hops on the number line from 0 to n. If he is on position i, then he can either hop to $i+1$ or to $i+2$. Let F_n be the number of ways he can get from 0 to n. Show that $F_n = F_{n-1} + F_{n-2}$.

2. How many diagonals does a regular n-gon have?

3. n letters are put in n addressed envelopes. Let d_n be the number of ways this can be done such that no letter goes into its designated envelope. Show that $d_n = d_{n-1} + (n-1)d_{n-2}$.

4. L is a set of n lines in $\mathbb{R}^2$. P is a set of n points in $\mathbb{R}^2$. For each point $p \in P$, let $d(p)$ be the number of lines in L passing through p. Show that

$$\sum_{p \in P} \binom{d(p)}{2} \leq \binom{n}{2}.$$

Bonus: Use this to show that

$$\sum_{p \in P} d(p) \leq \frac{n + n\sqrt{4n-3}}{2}.$$

5. Prove that

$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n}{i} = \binom{3n}{n}.$$

6. By interpreting both sides of the equation as "the number of ways of ____", show that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n.$$

7. How many positive integers less than 10000 are relatively prime to 10000?

8. How many positive integers less than 10000 are relatively prime to 30?

9. Show that

$$n! = n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n \ldots + (-1)^i \binom{n}{i}(n-i)^n + \ldots + (-1)^n \binom{n}{n}(n-n)^n.$$

10. Show that

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \ldots \binom{n+k}{k} = \binom{n+k+1}{k}.$$

(You may either do this directly using what you know about binomial coefficients, or by 2-way counting).

11. How many n letter strings can you form that uses each of the 26 letters of the alphabet at least once?