Problems

1. For a square matrix A, define $\sin A$ by the power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{matrix} 1 & 1996 \\ 0 & 1 \end{matrix}.$$

2. Let G be a group with identity e and let $\phi : G \to G$ be a function such that $\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$ whenever

$$g_1 g_2 g_3 = e = h_1 h_2 h_3.$$

Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e., for all x, y, we have $\psi(xy) = \psi(x)\psi(y)$).

3. If A and B are square matrices of the same size such that $ABAB = 0$, then must it be the case that $BABA = 0$?

4. Let S be a set of real numbers which is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three elements of T is in T, and that the product of any three elements of U is in U, show that at least one of the sets T, U is closed under multiplication.

5. Let G be a finite set of real $n \times n$ matrices $M_1, ...M_r$ which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} Tr(M_i) = 0$, where Tr denotes the trace (sum of the main diagonal). Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

6. Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence $g_1, \ldots, g_{2n}$ such that every element of G occurs exactly twice, and g_{i+1} equals $g_i a$ or $g_i b$ for each $i = 1, \ldots, 2n-1$, and $g_1 = g_{2n}a$ or $g_{2n}b$.

7. Let G be a finite group with identity e. If g and h are two elements in G such that $g^3 = e$ and $ghg^{-1} = h^2$, then find the order of h.

8. Suppose x, y, z are real numbers with $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$. Show that each of x, y, z lies in the interval $[2/3, 2]$. Can x attain the extreme values $2/3$ and 2?