

Problems

1. Let a, b, c be real numbers in $(0, 1)$. Suppose $a + b + c = 2$. Show that

$$\frac{abc}{(1-a)(1-b)(1-c)} \geq 8.$$

2. Let A_1, \dots, A_n be a regular n -gon which is inscribed in a circle of radius 1. Show that

$$\prod_{j=2}^n |A_1 A_j| = n.$$

(Hint: the polynomial $X^n - 1$.)

3. Show that there is an uncountable family of subsets of \mathbb{N} such that every two subsets in the family intersect in a finite set.
4. Let $a_1 = 1$, and define $a_n = a_{n-1} + \frac{1}{a_{n-1}}$ for each $n > 1$. Show that $\sqrt{2n-1} \leq a_n \leq \sqrt{3n-2}$.
5. An ant crawls at the rate of 10cm per minute along a rubber band which can be stretched uniformly. Suppose that the rubber band is initially one meter long and it is stretched an additional meter at the end of each minute. If the ant begins at one end of the band, does it reach the other end?
6. Let A be a set of 16 positive integers, each two being relatively prime to each other. Show that A has two elements whose product is greater than 2000.
7. What is the largest 3-digit prime factor of $\binom{2000}{1000}$.
8. Find the remainder when 19^{92} is divided by 92.
9. We have two positive integers bigger than 1, and suppose their sum is at most 100. S and P are heavy-duty mathematicians. Suppose S is given the sum of the integers and P is given the product of the integers.

They have the following conversation:

- P: I do not know the two numbers.
- S: I knew that you didn't know the two numbers.
- P: Now I know the two numbers.
- S: Now I know the two numbers.

What are the two numbers?

10. Suppose a, b, c are positive integers with no common factors. Suppose $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Show that $a + b$ is a perfect square.
11. Show that if $2^m - 1$ divides $2^n - 1$, then m divides n .
12. Given 7 arbitrary real numbers, show that there must be some two of them x, y such that

$$0 < \frac{x-y}{1+xy} < \frac{1}{\sqrt{3}}.$$

13. An octagon inscribed in a circle has side lengths a, a, a, a, b, b, b, b respectively. What is the radius of the circle in terms of a and b ?