MATH 642:491–Fall, 2012

Problems on 2-way counting and inclusion-exclusion

1. A frog hops on the number line from 0 to n. If he is on position i, then he can either hop to $i+1$ or to $i+2$. Let F_n be the number of ways he can get from 0 to n. Show that $F_n = F_{n-1} + F_{n-2}$.

2. How many diagonals does a regular n-gon have?

3. n letters are put in n addressed envelopes. Let d_n be the number of ways this can be done such that no letter goes into its designated envelope. Show that $d_n = d_{n-1} + (n-1)d_{n-2}$.

4. L is a set of n lines in $\mathbb{R}^2$. P is a set of n points in $\mathbb{R}^2$. For each point $p \in P$, let $d(p)$ be the number of lines in L passing through p. Show that
$$\sum_{p\in P} \binom{d(p)}{2} \leq \binom{n}{2}.$$
Bonus: Use this to show that
$$\sum_{p\in P} d(p) \leq \frac{n + n\sqrt{4n-3}}{2}.$$

5. Prove that
$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n}{i} = \binom{3n}{n}.$$

6. By interpreting both sides of the equation as "the number of ways of _____", show that
$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n.$$

7. How many positive integers less than 10000 are relatively prime to 10000?

8. How many positive integers less than 10000 are relatively prime to 30?

9. Show that
$$n! = n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n \ldots + (-1)^i \binom{n}{i}(n-i)^n + \ldots + (-1)^n \binom{n}{n}(n-n)^n.$$

10. Show that
$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \ldots \binom{n+k}{k} = \binom{n+k+1}{k}.$$
(You may either do this directly using what you know about binomial coefficients, or by 2-way counting).

11. A strong password needs to have at least one digit, at least one lower case alphabet and at least one upper case alphabet. How many strong passwords of length 10 are there?

12. How many n letter strings can you form that uses each of the 26 letters of the alphabet at least once?