The Pigeon-Hole Principle

(Note: The book has solutions to some of these problems–don't read them!)

1. Let A be a subset of integers of size n. Prove that there is a nonempty subset of A whose sum is divisible by n.
2. Given 19 distinct integers from the arithmetic progression $1, 4, 7, \ldots, 100$, prove that there must be two distinct integers that sum to 104.
3. Let q be an odd integer greater than 1. Show that there is an integer n such that q divides $2^n - 1$.
4. Let $A = \{1, 2, 3, \ldots, 100\}$. Let $B \subseteq A$ be such that for all distinct $x, y \in B$, $x + y$ is not divisible by 11. Show that $|B| \leq 47$.
5. Let A be a subset of size $n + 1$ consisting of positive integers in the range 1 to $2n$. Prove that there must be distinct elements a, b of A such that a is a divisor of b.
6. For any positive integer n, if S is a set of $2^n + 1$ points $\mathbb{R}^n$ with integer coordinates. Then there exists two of the points such that the midpoint of the segment between them has all integer coordinates.
7. Suppose we have 25 points inside a regular hexagon of side-length 2. Show that some two of them are within distance 1 of each other.
8. Let X be a real number and n a positive integer. Prove that at least one of the numbers $X, 2X, ..., nX$ is within $1/(n + 1)$ of an integer.
9. Let A be a finite subset of positive integers of size n Let $a_1, a_2, \ldots, a_t$ be a sequence of integers each belonging to A. Prove that if $t \geq 2^n$ then there are integers j, k satisfying $1 \leq j \leq k \leq n$ such that $\prod_{i=j}^{k} a_i$ is a perfect square.
10. Suppose S is a subset of $\{1, 2, \ldots, 2n + 1\}$ such that for any two distinct elements $a, b \in S$, their sum $a + b$ is *not* in S. Show that $|S| \leq n + 1$.
11. Let M be a matrix of real numbers, with each row in nondecreasing order. Suppose we sort each column into nondecreasing order. Prove that the rows are still in nondecreasing order.
12. Suppose 6 circles have a point in common. Prove that one of the circles contains the center of another circle.
13. Let B be a subset of $\{-1, 1\}^n$ (the set of points in $\mathbb{R}^n$ with all coordinates equal to -1 or $+1$). If $|B| > 2^{n+1}/n$, prove that B contains a set of three points that are the vertices of an equilateral triangle.
14. Let m, n be positive integers. Suppose $x_1, \ldots, x_m$ are positive integers between 1 and n and $y_1, \ldots, y_n$ are positive integers between 1 and m. Prove that there is a nonempty subsequence of consecutive entries of $x_1, \ldots, x_m$ and a nonempty subsequence of consecutive entries of $y_1, \ldots, y_n$ that have the same sum.

15. (Somewhat difficult) Let A, B be integer 2 by 2 matrices. Suppose that each of the matrices $A, A+B, A+2B, A+3B, A+4B$ has the property that it is invertible and its inverse has integer entries. Prove that $A+5B$ has the same property.